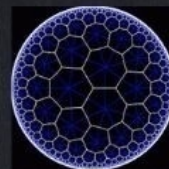




Blackboard Issue 8



$$\sum_n \frac{(-1)^n}{n^n} = \int_0^1 x^x dx$$

MTA (I)

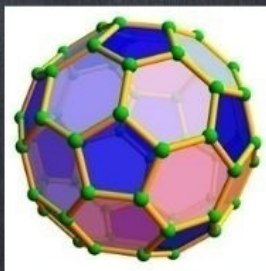
$$\sum_n \frac{1}{n^n} = \int_0^1 \frac{dx}{x^x}$$

Here is the Ramanujan-Hardy formula for the calculation of the number of partitions:

$$p(n) = \frac{1}{2\sqrt{2}} \sum_{k=1}^n \sqrt{k} A_k(n) \frac{d}{dn} \exp \left(\pi \sqrt{\frac{2}{3}} \sqrt{n - \frac{1}{24}} \right)$$

where

$$A_k(n) = \sum_{0 \leq m \leq k; (m, k) = 1} e^{\pi i [s(m, k) - \frac{1}{2} 2nm]}.$$



$$\sum_n n^n \int_0 x^x$$

$$e^{\pi\sqrt{163}} = 262537412640768743.9999999999992\dots$$

Examples

$$135^2 + 138^2 = 172^2 - 1$$

$$11161^2 + 11468^2 = 14758^2 + 1$$

$$7^3 + 10^3 = 12^3 + 1$$

$$6^3 + 8^3 = 7^3 - 1$$

$$791^3 + 812^3 = 1010^3 - 1$$

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{6 + \frac{1}{1 + \frac{1}{8 + \frac{1}{1 + \frac{1}{10 + \dots}}}}}}}}}}$$

22	12	18	87	22	12	18	87	22	12	18	87	22	12	18	87
88	17	9	25	88	17	9	25	88	17	9	25	88	17	9	25
10	24	89	16	10	24	89	16	10	24	89	16	10	24	89	16
19	86	23	11	19	86	23	11	19	86	23	11	19	86	23	11

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}} & 3\mathbb{Z} \oplus 2\mathbb{Z} & \longrightarrow & \\
 \downarrow & & \downarrow \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} & & \downarrow & & \\
 \longrightarrow & \mathbb{B}_d^a & \longrightarrow & \mathbb{Z}^4 & \xrightarrow{\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} & \mathbb{Z}^2 & \longrightarrow
 \end{array}$$

Editorial Board

Aekta Aggarwal (IIM Indore)
Anisa Chorwadwala (IISER Pune)
Sangeeta Gulati (Sanskriti School, Delhi)
Neena Gupta (ISI Kolkata)
Amber Habib (Shiv Nadar Institution of Eminence, Delhi NCR)
S Kesavan (Formerly IMSc, Chennai)
Manjunath Krishnapur (IISc, Bangalore)
Anupam Saikia (IIT Guwahati)
Shailesh Shirali (Sahyadri School KFI, Pune)
B Sury (ISI Bangalore): Editor-in-Chief
Geetha Venkataraman (Dr B R Ambedkar University Delhi)
Jugal Verma (IIT Gandhinagar)

Advisory Board

S G Dani (Mumbai)
R Ramanujam (Chennai)
V Srinivas (Mumbai)
K Subramaniam (Mumbai)

The aim of *Blackboard*, the Bulletin of the Mathematics Teachers' Association (India), is to promote interest in mathematics at various levels and to facilitate teachers in providing a well-rounded mathematical education to their students, in curricular as well as extra-curricular aspects. The Bulletin also serves as an interface between MTA (I) and the broad mathematical community.

© Mathematics Teachers' Association (India)

Registered Office

Homi Bhabha Centre for Science Education
Tata Institute of Fundamental Research
V. N. Purav Marg, Mankhurd
Mumbai, 400088 INDIA

<https://www.mtai.org.in/bulletin>

Blackboard

Bulletin of the Mathematics Teachers' Association (India)
Issue 8

June 2025

Contents

Editorial	3
1 A Mathematician's Lament, book review by AmisH Parmar	5
2 Rubik's Cube – Mathematical Magic, by Geetha Venkataraman	9
3 Regularity in Prime Sided Polygons with Rotational Symmetry, by Rajku-mar Shirol	21
4 A Crazy Infinite Series Paradox, by Sai Dibyanshu	29
5 The Nine-point Circle of a Triangle – Part 3, by Shailesh Shirali	33
6 Repeating Coprime Integers Eventually Ending at 1, by Anand Prakash	41
7 The Champagne Problem – Distilled from Hilbert 17, by B. Sury	45
8 A Spiral Prism on a Regular Base, by Jyotirmoy Sarkar	51
9 Derksen's Proof of the Fundamental Theorem of Algebra, by J. K. Verma	67
10 To Identify, or not to Identify, that is the Question, by S. Kesavan	73

Editorial

The present times witness many mathematical developments happening simultaneously on many fronts. Every month, on the arXiv, some 15000 articles in mathematics, computer science and physics are posted. In Mathematics alone, there are 3000 to 4000 monthly submissions. This makes it difficult for even active researchers to have even a semblance of the state of the art except in her own narrow area. Teachers are understandably busy with such heavy teaching duties that they cannot practically acquaint themselves with a part of this huge influx of research papers. Thus, for the knowledge from these active databases to trickle down to the teachers, it takes a long time. Those who are fortunate enough to be in higher institutions where research is well supported, must make an effort to inform their teacher colleagues from other institutions as they do not have that luxury. Many teachers would like to be involved in research work in the limited time available to them. For this to happen, this trickling down of information mentioned above is of vital importance.

Blackboard makes a modest effort by attempting to publish easy to read expositions of important developments that have emerged over time and are continuing to emerge daily in the mathematical world. More often than not, such articles are not directly related to what the teachers use in their classrooms but give rise to a mathematical temper. In addition, Blackboard encourages teachers to write something of interest to the mathematical community. An advantage of mathematics as compared to some other disciplines is that one can ask questions that look interesting and non-obvious to answer irrespective of whether they are immediately seen to be applicable or not. Problem solving abilities could be developed through such exercises.

The present issue contains a potpourri of paper with diverse flavors. While a student has written on rotational symmetry and regular polygons, another one has described paradoxical aspects of considering infinite series. A riveting article on spiral prisms is accompanied by a thought-provoking one which is on duality in functional analysis but couched in the language of dualism in philosophy. Other interesting morsels include a discussion of the classical nine-point circle and a description of the symmetric intricacies of the Rubik cube. A lovely new proof of the fundamental theorem of algebra is aside an article on sums of squares of rational functions that arose in the context of Hilbert's 17th problem. Last but not least, there is a book review that should interest both teachers and their students; the book is entitled 'A Mathematician's Lament'. All in all, we hope that the issue has write-ups of interest pertaining to different tastes.

We also welcome Professor Manjunath Krishnapur to the editorial team of Blackboard.

In recent mathematical news, three mathematicians have shown that for large N , approximately 69 per cent of d -regular graphs on N vertices are so-called Ramanujan graphs; these are optimal expander graphs which means they have few vertices but are still highly connected.

There are many new developments that have occurred some of which address deep open problems. Those who are interested in looking at the statements of some deep open problems in mathematics, can find a list of 100 problems in Ben Green's webpage. The very first one has now been solved by a graduate student very recently.

Another exciting bit of news concerns Rajula Srivastava from Bonn who has been awarded the Maryam Mirzakhani New Frontiers Prize which is awarded to outstanding female mathematicians who have recently completed their doctorate. She has made progress in a challenging area at the interface between harmonic analysis and number theory. The award carries 50,000 dollars and she plans to donate part of the prize money to organizations in India that support children's education.

B. Sury

Indian Statistical Institute Bangalore

1 June, 2025

1 Book Review: ‘A Mathematician’s Lament’ by Paul Lockhart

AmisH Parmar

Seed2Sapling Education, Bangalore

Email: amishparmar19dec@gmail.com

Did you find your math class boring and hated it? Do you feel your time as a student was wasted in meaningless memorization of mathematical ‘facts’ and ‘representations’, which you certainly never used outside tests? Or perhaps you feel helpless as a math teacher, having your students ‘solve problems’ and would like them to engage in ‘problem solving’ instead? Then you need to experience this book by Paul Lockhart, a professional mathematician who feels exactly the same.

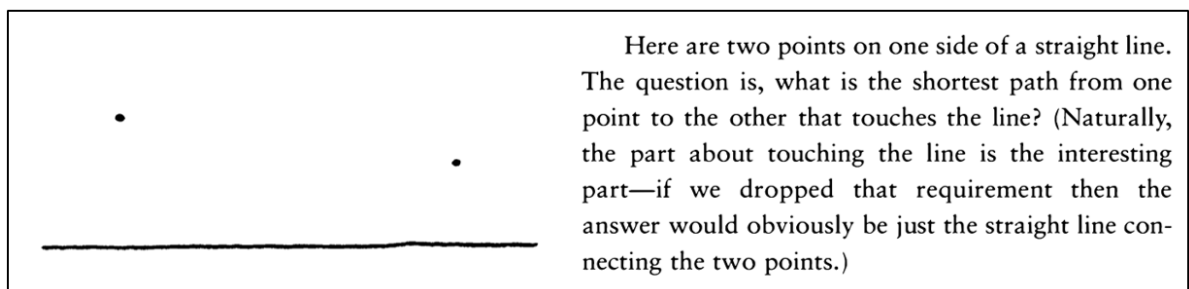
This strongly worded opinion piece found its humble origins in a 25-page essay written decades ago. Formalized as a book in 2009, *A Mathematician’s Lament* [PL] is not only a critique of how mathematics is typically taught in schools, but also a heartfelt and thought-provoking reflection by someone who deeply loves the subject. Despite being written in the context of the American educational system and published over 15 years ago, it remains just as relevant today for anyone who was ‘forced’ to study math or is concerned about the state of math education.

The book begins with a ‘musician’s nightmare’, a scenario that gradually unsettles the reader, as they realise the bitter truth of what they have been robbed of. What follows is Lockhart’s compelling argument of math being ‘the most fascinating & imaginative art form’ instead of the rigid science it’s portrayed to be. Throughout the chapter he draws convincing parallels between math and what the ‘society & culture’ considers ‘art’, arguing they are the same. In fact, thanks to its independence of physical limitations, he proclaims math to be the purest of all art forms, and yet it is reduced simply to some sort of ‘tool’ for science.

The subsequent chapter, titled ‘Mathematics in School’, starts with a discussion of the so-called ‘math reform’. Lockhart boldly criticizes both textbook publishers and educators for their approach to making math ‘friendly’—by inventing nonsensical stories to help students memorize formulas, creating contrived contextualizations and relevance

to daily life and passing dull, mechanical ‘exercises’ as math problems. He then addresses teaching, again analogizing with the art of painting, believing teachers should be creators of knowledge and not mere data transmitters. Next in line is the math curriculum itself, whose rigidity leads to unmotivated definitions and senseless distinctions, which end up presenting math as full of ‘rules & regulations’ instead of ‘explorations & discovery’. Even logical and beautiful proofs in geometry are not spared, falling prey to excessive notation and soul-less formalizing framework. In his signature sarcastic style, he ends his lament by briefing about the topics covered in school math, calling it “a proven cure for curiosity” and, as Simplicio (more on him later) rightfully claims, leaving us thoroughly depressed.

In the second part of the book – *Exultation*, Lockhart shares what math really is and why he loves it, in hopes to repair the damage caused by school education, and perhaps even inspiring the next generation of learners. He once again emphasises that, math being an art form, one needs to ‘do’ it! Beginning with the basics, here he compares mathematicians with field biologists (professions which we think are poles apart), studying how the exotic creatures we call ‘numbers’ ‘behave’. In the thoughtful act of personifying numbers, he aims to highlight the concepts of representations, abstractness and their intrinsic properties. He encourages ‘creation’ of mathematics, and adds- “We have the authority to define our creations, to instill in them whatever features or properties we choose, but we have no say in what behaviors may then ensue as a consequence of our choices.” Not just in this section, but throughout the book, Lockhart presents these snippets of mathematical problems, stressing their complete irrelevance to our lives, and precisely that is, he argues, what makes them even more fun and beautiful. Here, I would like to share one such problem which fascinated me the most, hoping the reader of this article would engage with it and have fun by doing so!



A snapshot of the shortest path problem from the book

Lockhart also touches upon my favourite topic in math – proofs, believing that one should engage with them, however frustrating they might seem, and they often manage to ‘prove more than we intended’, opening a segue to new explorations. He thinks of them as a “Two-Headed Monster”, where one head ‘demands a logically airtight explanation’ and the other wants to ‘see simple beauty and elegance’. This seeming dichotomy is not

just desirable but necessary – this way of thinking has personally helped me to not be satisfied with just $LHS = RHS$, but to also seek the underlying charm of the proof itself. He signs off by giving the only practical advice he has to offer – just play with math!

Lockhart's writing style is certainly peculiar – his vivid, fearless use of language when he writes “...*senseless, soul-crushing ideas that constitute contemporary mathematics education...destroying a child's natural curiosity and love of pattern-making...*”, or when he makes bold statements like “*Schools of education are a complete crock*”. Or be it his unfiltered lack of euphemism, like in “...*kill enthusiasm and interest in a subject (is to) make it a mandatory part of the school curriculum*”. One feels unprepared for such brutal honesty. Employing strongly expressive, colloquial phrases like ‘damn it!’ may seem excessive at first, but they all are a result of his genuine concern, frustration and deep thought about the state of mathematics education.

The end of chapters are written in dialogue style using the classic characters of Simplicio and Salviati from Galileo's popular work. With this, Lockhart manages to address almost every thought that might arise in the reader's mind, making one feel as if they are directly conversing with the author himself!

Reading this book somehow reminded me of the article titled “*Needed: a problem solving culture*” by R. Ramanujam, published in the very first issue of *Blackboard*. He too proposes that every student should engage in an enjoyable and exploratory mathematical activity every year, stressing on learning math techniques, like any art, in the context of playing with problems.

A Mathematician's Lament is an inspirational read not just for learners and teachers of mathematics, but for anyone ‘searching for love and meaning’, offering a ‘glimpse of something harmless, joyful & pure’. I profess it to be a must read for adults who have no need for math in their daily lives, who have been victims of the educational system, conditioning them to hold a major misconception about what mathematics truly is!

Bibliography

[PL] Paul Lockhart. *A Mathematician's Lament*. Bellevue Literary Press, 2009.



Amish Parmar is a Science Facilitator at Seed2Sapling Education, but also enjoys doing Math. After completing his Masters from Department of Physics, Savitribai Phule Pune University, he worked at Homi Bhabha Centre for Science Education in the Vigyan Pratibha project for 5 years. When he is not pondering over education, he enjoys reading, boardgaming, jigsaw puzzling & birding.

2 Rubik's Cube – Mathematical Magic

Geetha Venkataraman

School of Liberal Studies, Dr. B. R. Ambedkar University Delhi,
Kashmere Gate, Delhi 110006
Email: geetha@aud.ac.in

1 Introduction

The year was 1974 and Ernő Rubik was a professor of architecture teaching a class on descriptive geometry. He started experimenting with putting cubes together into a larger cube, so that they could move and exchange places. He hoped to use it as a prototype which could help with his teaching. From this tinkering was born the 'magic cube' which we know now as Rubik's cube. It has mesmerised generations in its fifty year existence. Coincidentally, Ernő turns 80, in this year of 2024. Below is a picture of the Rubik's cube.

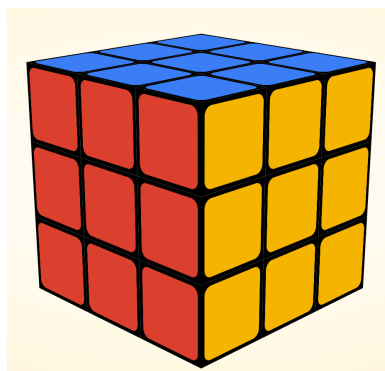


Figure 1: Rubik's cube solved

As can be seen from the picture, each of the six faces of the cube is made up of nine smaller cubes. The central cube in each face remains fixed. In its starting or **solved**

position each face of the Rubik's cube has a single colour. In Figure 1, three such faces are shown coloured blue, yellow and red, respectively.

You can also imagine the Rubik's Cube as being made of three slices, top, middle (h) and bottom (or left, middle (v) and right). The (h) and (v), indicate the middle slice when the Rubik's cube is sliced horizontally versus when it is sliced vertically. Note that each slice is made up of nine cubes with the central cube being fixed. Further, each of these slices rotates about the central fixed cube. This is irrespective of which of the six faces is at the top. Once several such rotations of each slice is performed the monochrome faces from the starting position get scrambled. The picture below shows the left slice being rotated slightly after some scrambling of the faces has already taken place.

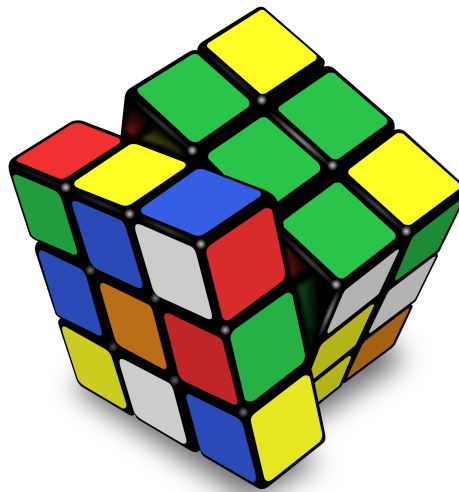


Figure 2: Rubik's cube scrambled

The puzzle of course is to start with a scrambled Rubik's cube as in Figure 3. The aim then is to use a sequence of rotations of the slices; either of top, middle (h), bottom or left, middle (v), right, to bring the cube back to the solved position with each of the six faces being of a single colour respectively.

How many different scrambled versions can you have of the cube? Given any scrambled Rubik's cube, can we get back to solved position? If so, what is the least number of moves that are required to get back to the scrambled position? To understand such questions and to seek some answers we turn to mathematics.

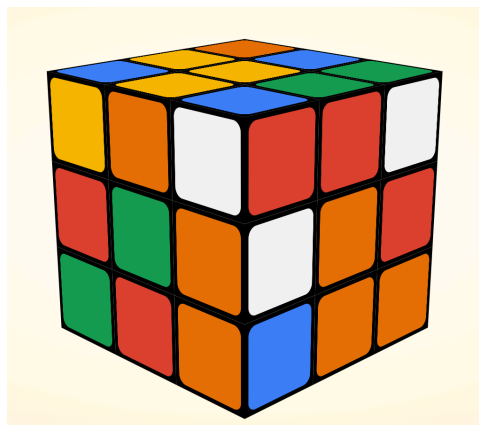


Figure 3: Rubik's cube puzzle

2 Symmetry and Groups

Before we can study the Rubik's cube, wearing mathematical spectacles, we need to have some basic understanding of symmetry from a mathematical view point and to learn about certain algebraic objects called groups that measure symmetry. We are all of course intuitively aware of what it means for any object to be symmetrical. Let us extend this intuitive understanding to a working definition of symmetry.

Given any object in our real world, we can think of it as occupying a fixed position in the three dimensional space we live in. Let us think of it as living in a hole in space which is exactly the same shape as the object. A symmetry of the object is an action that you can perform on that object to bring it back to occupying the imaginary fixed object shaped hole looking exactly as it did before the action. In other words, if an audience that was watching you closed their eyes when you performed this action, they would not know that anything had been done to the object. A caveat is that this action needs to be performed on the object as a whole and one is not allowed to break the object and reassemble it.

An easier experiment that any of us can perform is to make a cut out of a square using slightly thick paper and then to draw the outline of this square on a sheet of paper. Now think of the actions that we could perform on the square cut-out so that it comes back and occupies the outline drawn, after the action. Note that you are not allowed to tear and then reconstruct the square. It is not difficult to see that there are precisely **eight** such actions that can be performed. These are the **four rotations** of 0, 90, 180 and 270 degrees respectively about an axis perpendicular to the plane of the paper and passing through the centre (point of intersection of the diagonals) of the square cut-out. The other **four are reflections** about lines joining the mid-points of opposite sides

and about the two diagonals. The figure below illustrates these eight symmetries of a square. In order to keep track of the actions or symmetries we mark the vertices of the square (ABCD) in an anti-clockwise direction and imprint the same on the outline corresponding to the starting position of the cut-out.

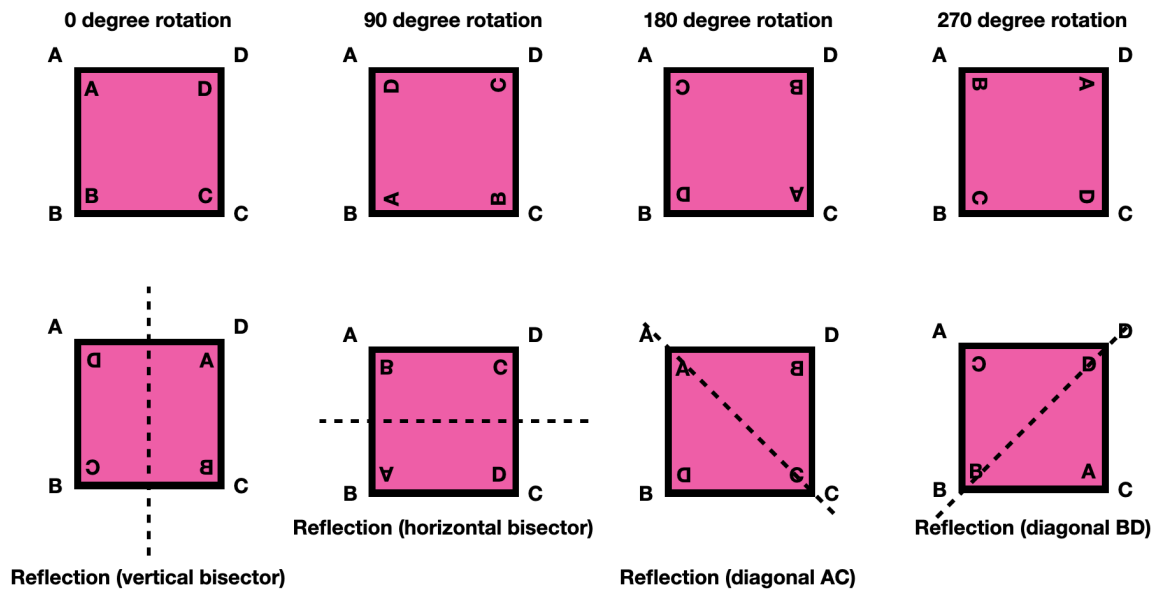


Figure 4: Symmetries of a Square

Can you figure out a pattern with the symmetries of any regular n -gon (namely an n -sided figure with equal sides and equal interior angles)? It turns out that such a regular n -gon has exactly $2n$ symmetries, n rotations and n reflections. Figure 5 shows the eight lines of reflection for a regular octagon. It has a basic rotation of 45 degrees. So the other non-zero rotational symmetries will be multiples of 45 degrees until 315 degrees.

Formally a symmetry of an object X is a distance preserving bijective function from X to itself. Each of the eight symmetries in Figure 4, can be represented as bijective functions (permutations) mapping the set of vertices $\{A, B, C, D\}$ back to itself. This is because any symmetry of the square permutes the vertices amongst themselves. These are expressed below in what is known as a cycle representation for permutations.

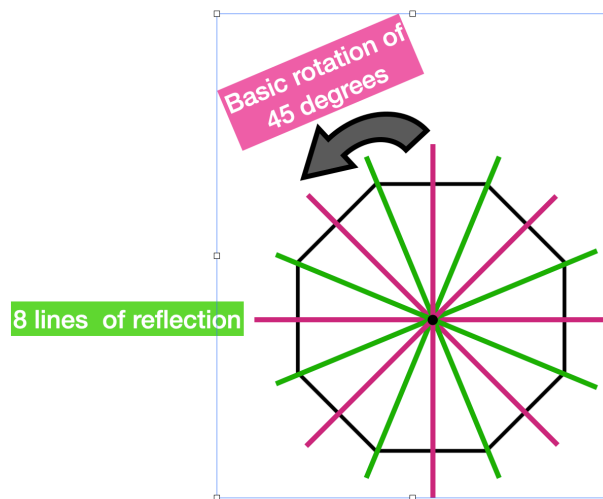


Figure 5: Reflections and Rotations of a regular Octagon

The four rotations R_0, R_{90}, R_{180} and R_{270} are given below as permutations in the cycle representation.

$$R_0 = (A)(B)(C)(D), R_{90} = (A, B, C, D), R_{180} = (A, C)(B, D), R_{270} = (A, D, C, B).$$

The notation $(A, C)(B, D)$ indicates that the vertex A goes to C and vice-versa under the 180 degrees rotation and that vertex B gets mapped to D and vice versa. Similarly R_{270} moves vertex A to D , the vertex D to C , the vertex C to B and finally B to A .

Now let us write down the four reflections H_v , the reflection about the vertical bisector, H_h , the reflection about the horizontal bisector, H_d , the reflection about the diagonal AC and finally $H_{d'}$, the reflection about the diagonal BD below.

$$H_v = (A, D)(B, C), H_h = (A, B)(C, D), H_d = (A)(B, D)(C), H_{d'} = (A, C)(B)(D).$$

Let X be an object in our three-dimensional space. Let $\text{Sym}(X)$ denote the set of all symmetries of X , namely the bijective functions (permutations of X) that also preserve distance. Just using the working definition of symmetry, we see that if f and g are symmetries of X and $f \circ g$ denotes the action of g on X followed by f , then it is obvious that a symmetry followed by a symmetry, namely $f \circ g$ is again a symmetry of X . This property is called **closure**. See Figure 6.

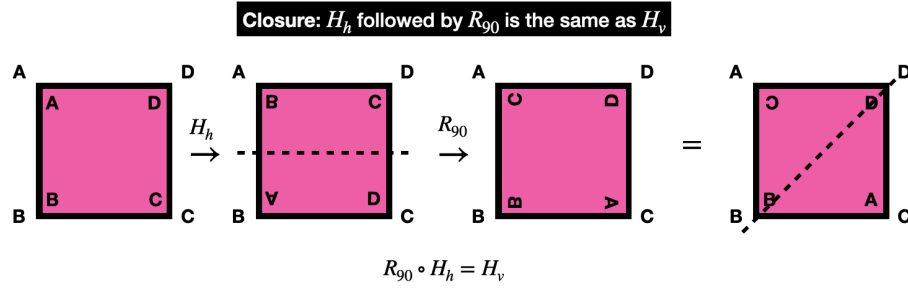


Figure 6: Illustration of Closure

Similarly it is not difficult to check that for $f, g, h \in \text{Sym}(X)$ we have $(f \circ g) \circ h = f \circ (g \circ h)$ or in other words $\circ$ or composing symmetries is **associative**. Also note that the 0 degree rotation, which is always a symmetry of X and is nothing but the identity function on X , namely $I_X : X \rightarrow X$ as $I_X(x) = x$ for all $x \in X$. Further, $f \circ I_X = f = I_X \circ f$ for all $f \in \text{Sym}(X)$. Thus there exists $I_X \in \text{Sym}(X)$ which is an **identity** with respect to composition. Finally, its easy to check that for each $f \in \text{Sym}(X)$ there exists another symmetry $f' \in \text{Sym}(X)$ such that $f \circ f' = I_X = f' \circ f$. In other words every symmetry has an **inverse** in $\text{Sym}(X)$. See Figure 7 for an illustration that the inverse of R_{90} is R_{270} .

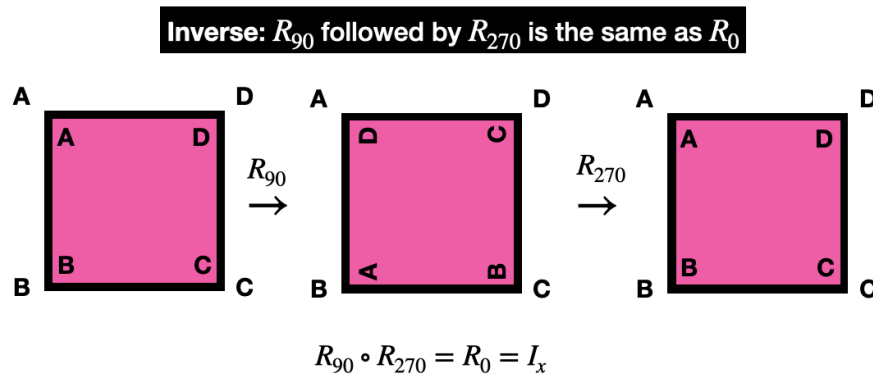


Figure 7: Illustration of existence of inverse

The properties of closure, associativity, existence of identity and inverses makes $\text{Sym}(X)$ a **group** with respect to the operation of composition ($\circ$). In a group, it can be shown that the identity and inverses are unique. The inverse of the symmetry f will be denoted as f^{-1} .

So if X is a square then $\text{Sym}(X) = \{R_0, R_{90}, R_{180}, R_{270}, H_v, H_h, H_d, H_{d'}\}$ is the group

of symmetries of the square. It is an interesting exercise to identify the inverses of the symmetries of a square.

In general a **group** G is a **non-empty set** with an **operation** $*$ (a good way of combining any two elements of G , much like composition of symmetries or addition on integers) which is **closed**, that is, $g * h \in G$ for all $g, h \in G$; **associative**, that is $(g * h) * k = g * (h * k)$ for all $g, h, k \in G$; has an **identity element** $e \in G$, that is, $g * e = g = e * g$ for all $g \in G$ and every element g of G has an **inverse** $g^{-1} \in G$ such that $g * g^{-1} = e = g^{-1} * g$. As seen above the set of symmetries of any object X is a group under composition. Other familiar examples are the set of integers under addition or the non-zero rational numbers under multiplication. The latter two groups are also **abelian** as $g * h = h * g$ for all g, h in those groups. See Figure 8 which shows the properties satisfied by addition on the set of integers making it into a group.

Let a, b and c be integers. Then we have

Closure:
 $a + b$ is an integer, for example
 $-5 + 3 = -2$.

Associativity:
 $(a + b) + c = a + (b + c)$, for example
 $(-5 + 3) + -2 = -2 + -2 = -4$
 $-5 + (3 + -2) = -5 + 1 = -4$

Existence of identity:
 $a + 0 = a = 0 + a$, for example
 $6 + 0 = 6 = 0 + 6$

Existence of inverses:
 $a + (-a) = 0 = (-a) + a$, for example
 $3 + (-3) = 0 = (-3) + 3$

Figure 8: Addition on the set of Integers

Another important group that can be defined on any non-empty set X is the **group of permutations** of X . This is denoted as S_X . Thus

$$S_X = \{f : X \rightarrow X \mid f \text{ is a bijective function on } X\}$$

and the operation is $\circ$, namely, composition of functions. Note that if X is an object in three-dimensional space then $\text{Sym}(X) \subseteq S_X$ and both are groups under the same operation of composition. We therefore say that $\text{Sym}(X)$ is a **subgroup** of S_X and write $\text{Sym}(X) \leq S_X$. Another example is that the set of integers under addition is a

subgroup of the set of rational numbers under addition. When X is a finite set with $|X| = n$, we write S_n instead of S_X . Further, S_n has $n! = n(n-1) \cdots 1$ number of elements. Thus, when X is a square we get that $\text{Sym}(X) \leq S_4$. Note that only 8 of the 24 permutations of S_4 are symmetries of a square. Here we treat X as being represented by its four vertices. In the case when X is an equilateral triangle (represented by its three vertices) we in fact get that $\text{Sym}(X) = S_3$.

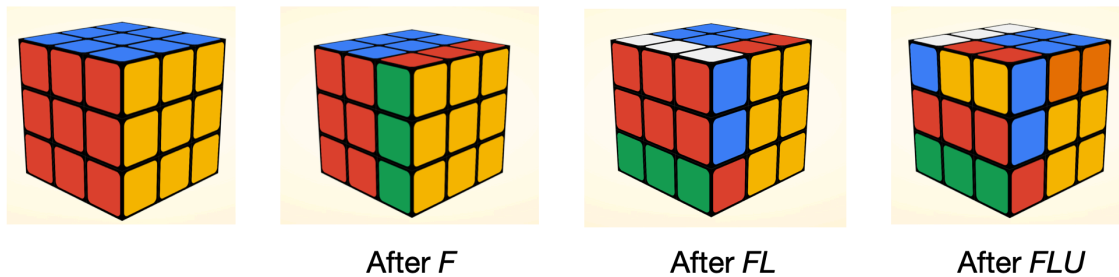
It is not difficult to see that the more symmetrical an object, the larger its group of symmetries. If we consider X to be a triangle then $\text{Sym}(X)$ will be the largest of order 6 when X is an equilateral triangle. Whereas it has order 1 if X is scalene and $|\text{Sym}(X)| = 2$ when X is an isosceles triangle which is not equilateral. So the group of symmetries or the **symmetry group** of an object is a measure of how symmetrical the object is. So far, the examples we have considered have been two dimensional objects.

Finding the symmetries of a cube requires much more work! However, the solved Rubik's cube only has the trivial or 0 degree rotation as its only symmetry as each face of the solved Rubik's cube has a different colour. Instead we turn our attention to a group that arises out of basic moves that can be made on the Rubik's cube (scrambling of the cube). This group is called the **Rubik's cube group**.

3 The Rubik's Cube Group

Let us now explore the mathematics of the Rubik's cube in detail armed with some knowledge of groups. As discussed in the first section, the Rubik's cube is made of three slices each containing 9 smaller cubes. So we have 27 cubes in all, with 26 of them being visible and the one at the centre of the Rubik's cube which is not visible. We also use a labelling of the six faces of the Rubik's cube using notation developed by David Singmaster. The labels for the faces are: right (r), left (l), up (u), down (d), front (f) and back (b). In Figure 1, the blue coloured face is u , the red face is l and the yellow face is f . The other three faces d , r and b are not visible.

We will also use capital letters associated with the respective faces to describe basic moves of the Rubik's cube. The letter R will denote a rotation of 90 degrees in the clockwise direction of the right face (clockwise when facing the right face). Similarly L, U, D, F, B will denote the 90 degrees clockwise rotations of the respective faces. A scrambled position of the Rubik's cube is achieved when we apply a finite sequence of any length whose elements are these six basic moves. Figure 9 shows the solved Rubik's cube from Figure 1 after the basic move of F, L and U are applied in a sequence.

Figure 9: Move FLU

Note that any move of the Rubik's cube or alternatively any scrambled version of the Rubik's cube can be achieved by a finite sequence of some length consisting of the letters R, L, U, D, F, B . Further, each of these six basic moves can be reversed. For example, rotating the right face by 90 degrees in the anti-clockwise direction reverses R and is nothing but R applied thrice or the sequence RRR . We can denote this by R^{-1} . Thus the scrambling achieved by the FLU move can be reversed by the move $U^{-1}L^{-1}F^{-1}$ and the Rubik's cube will return to the solved position. Now we are in a position to describe the Rubik's cube group as a subgroup.

We see that any basic move of the Rubik's permutes the 54 smaller squares that make up the six faces of the Rubik's cube. So each of the basic moves can be regarded as a member of S_{54} . Further since every scrambled version of the Rubik's cube is made up of a finite sequence of some length in which the letters from R, L, U, D, F, B occur, we can define the Rubik's cube group to be the intersection of all the subgroups of S_{54} that contain the subset $\mathcal{B} = \{R, L, U, D, F, B\}$ of the basic moves. This intersection turns out to be the smallest subgroup of S_{54} containing $\mathcal{B}$. Let us denote the **Rubik's cube group** by $\mathcal{G}_{RC}$. Then $\mathcal{G}_{RC} \leq S_{54}$ and

$$\mathcal{G}_{RC} = \{a_1 a_2 \dots a_n \mid n \text{ is a natural number and } a_i \in \mathcal{B} \text{ for all } i = 1, \dots, n\}.$$

We also say that $\mathcal{G}_{RC}$ is the subgroup of S_{54} generated by the subset $\mathcal{B}$ and write $\mathcal{G}_{RC} = \langle \mathcal{B} \rangle$.

Now let us use the above group to find out the number of scrambled versions we can have of the Rubik's cube. Clearly each scrambled version of the Rubik's cube corresponds to an element of the Rubik's cube group $\mathcal{G}_{RC}$. So we just need to find the order or the number of elements in this group. We label the positions on the Rubik's cube using the labelling of the faces. See Figure 10 which shows 7 of the visible corner positions labelled. The eighth position which is not visible drb . Similarly we can label the mid-edge positions as well as the centre positions on each face. An illustration of one each is included.



Figure 10: Labelling of positions

We note that each of the six basic moves keeps the central small cube of its face fixed. Thus any move of the Rubik's cube and hence any element of $\mathcal{G}_{RC}$ keeps the central small cubes of each face fixed. Further the corner small cubes (8 of these) are permuted amongst themselves by any basic move and hence any move. The mid-edge small cubes (12 of these) are also permuted amongst themselves by any basic move and hence any move. This is illustrated by Figure 11 which shows the movement of the centre small cube in f position, corner small cube in the ulf position and the mid-edge small cube in the rf position after the basic move F is applied.

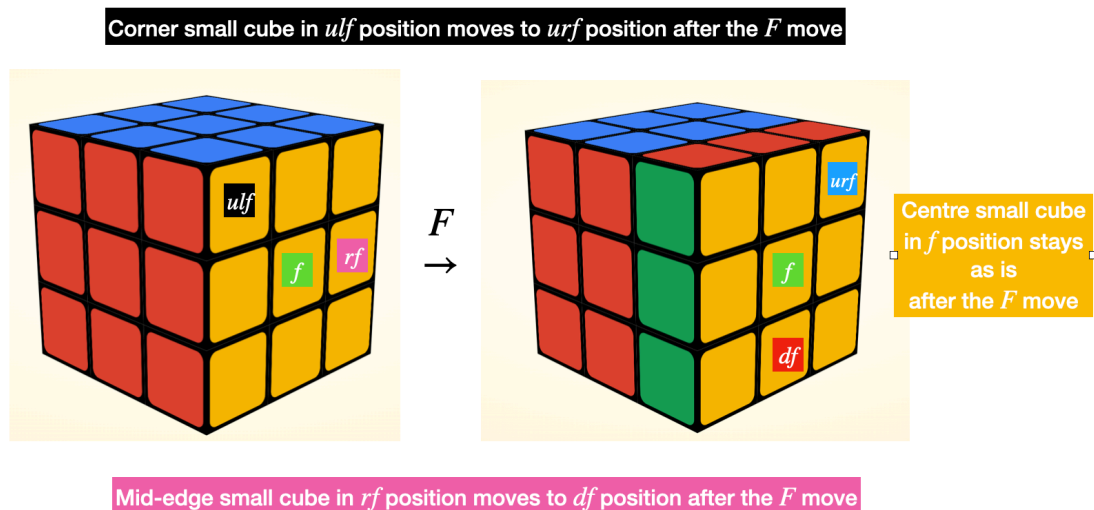


Figure 11: Moves of the centre small cube, the corner small cube and the mid-edge small cube

Also note that any corner small cube can occupy any of the 8 corner positions by a choice of suitable moves and similarly, any of the mid-edge small cubes can occupy any of the 12 mid-edge positions. Using the above facts we can find the number of scrambled configurations of the Rubik's cube. Consider the corner position '*ulf*' of the Rubik's cube. We have 8 choices to fill it with a corner small cube. Once a corner small cube is fixed in this position we consider another corner position say '*ulb*'. For this we will have 7 choices. Thus we have $8! = 8 \cdot 7 \cdots 2 \cdot 1$ possible ways of positioning the 8 corner small cubes in the 8 corner positions available. From Figure 1, we note that the corner small cube in position '*ulf*' has blue colour on the *u* face, red colour on the *l* face and yellow colour on the *f* face. But again it's possible for any of these three colours to be on the *u* face and this will automatically decide the colour of the other two faces. Thus each corner small cube can lie in a fixed corner position in 3 different ways and since there are 8 such corner small cubes we have 3^8 ways in all that they can be oriented in the corner positions. Therefore there are $3^8 \cdot 8!$ possible **configurations for the corner small cubes**. Similarly we will have $2^{12} \cdot 12!$ possible **configurations for the mid-edge small cubes**. Thus in all we have $3^8 \cdot 8! \cdot 2^{12} \cdot 12!$ possible **configurations of the Rubik's cube**. This number is approximately **519 quintillion**.

However, a point to be noted is that while these are all the possible permutations of the corner small cubes and the mid-edge small cubes, not all of these maybe achievable using the moves of the Rubik's cube. This is much like the fact that only 8 of the 24 permutations turn out to be symmetries of a square. A particular configuration of the Rubik's cube as described above is a valid one (or an element of the group $\mathcal{G}_{RC}$), only if we can achieve that configuration from the solved position using moves of the Rubik's cube or alternatively we can use the moves of the Rubik's cube to get back to the solved position from the given configuration.

We need a lot more mathematical knowledge as well as knowledge of groups and group actions to find out how many of the above Rubik's cube configurations are valid. It turns out that that only a twelfth of the above configurations are valid. In other words

$$|\mathcal{G}_{RC}| = \frac{1}{12} \cdot 3^8 \cdot 8! \cdot 2^{12} \cdot 12! = 3^7 \cdot 8! \cdot 2^{10} \cdot 12!.$$

A note on the figures

The image for Figure 2 has been taken from Wikimedia Commons and all the details can be found at https://en.m.wikipedia.org/wiki/File:Rubik%27s_cube_v3.svg. The other figures related to the Rubik's cube have been created by the author using the virtual Rubik's cube available at [VRC].

Acknowledgement

This article was written for the public exhibition on the Rubik's Cube held by the Liszt Institute Delhi - The Hungarian Cultural Centre, at the Rajiv Gandhi Science Centre in Delhi from 26 July 2024. The exhibition was held to celebrate the 50th anniversary of the Rubik's Cube and the 80th birthday of its inventor, Hungarian architect, Ernő Rubik.

Bibliography

- [AA] Alexander Alter. *He Invented the Rubik's Cube. He's Still Learning From It*. New York Times. September 16, 2020. <https://www.nytimes.com/2020/09/16/books/erno-rubik-rubiks-cube-inventor-cubed.html>.
- [JC] Janet Chen. *Group Theory and the Rubiks Cube*. Lecture Notes, Harvard University. <https://people.math.harvard.edu/~jjchen/docs/Group%20Theory%20and%20the%20Rubik's%20Cube.pdf>.
- [NST] Peter M. Neumann, Gabrielle A. Stoy, and Edward C. Thompson. 1994. *Groups and Geometry*. Oxford Universities Press. <https://global.oup.com/academic/product/groups-and-geometry-9780198534518?cc=in&lang=en&>.
- [VRC] Virtual Rubik's Cube. <https://rubikscu.be/>.



Geetha Venkataraman is professor of mathematics at Dr. B. R. Ambedkar University Delhi. Her area of research is finite group theory. Apart from her interest in group theory and related areas, she is deeply interested in popularising mathematics, mathematics education, issues related to women in mathematics and women in leadership in academia.

3 Regularity in Prime Sided Polygons with Rotational Symmetry

Rajkumar Shirol

Azim Premji University, Bengaluru, India

Email: rajkumar.shirol21ug@apu.edu.in

Abstract

We prove that a polygon with a prime number of sides must be regular if it has a non-trivial rotational symmetry. Although this result can be easily shown using group theory, we provide a proof based solely on elementary geometry.

1 Introduction

When a figure looks identical when rotated, it is said to exhibit rotational symmetry. Rotational symmetry can be visualised both in plane and space. In this paper, we will prove a special property of polygons which satisfy rotational symmetry. The following theorem states this property.

Theorem 1.1. *Any polygon with prime number of sides and rotational symmetry with order > 1 , is a regular polygon.*

Proving this theorem involves examination of periodicity of side lengths and angles in polygons which satisfy rotational symmetry. To make the discussions clearer, the paper states some preliminary definitions and results.

The reason for adopting a non-group theory approach is to make it accessible to high school students who are not yet familiar with groups. The results proved in this paper can be used further in the study of area, perimeter of polygons with rotational symmetry, in engineering, rotational dynamics of objects.

We first describe some preliminary terminology related to rotational symmetry in the next section. In the last section, we explain the proof idea and the proof.

2 Preliminaries

Definition 2.1. A planar figure F has rotational symmetry if there exists a point O and an angle ϕ , such that rotating F about O by ϕ degrees anticlockwise, results in an identical figure. The point O is called a **center of rotation**.

Remark 2.2. The terms “identical” and “congruent” have distinct meanings. A figure F is identical to F' if it coincides with F' without any additional rigid transformations. A figure F is congruent to F' if it coincides with F' after required rigid transformation. While identical figures are always congruent, the converse is not necessarily true.

Example 2.3.

1. A **circle** has rotational symmetry about its center for any angle.
2. An **equilateral triangle** has rotational symmetry about its centroid for any multiple of 120° .
3. A **line segment** has rotational symmetry about its midpoint by 180° .

Now, we move forward by listing down some basic definitions and results related to rotational symmetry.

- If a plane figure F has rotational symmetry (about O) by an angle α , it also has rotational symmetry (about O) for any positive multiple of α .
- Let a planar figure F have a rotational symmetry about a point O by an angle α . The smallest possible non-zero angle for which this symmetry holds is called the **smallest angle of rotation**. We denote this angle by θ from now on.
- If rotational symmetry holds for a figure at an angle $\alpha > 0$, then there exists a positive integer n such that $\alpha = n\theta$. This can be easily shown using division algorithm.
- In any figure with rotational symmetry with the smallest angle of rotation θ , there exists a positive integer n such that $n \times \theta = 360^\circ$. This number n is called the

order of rotational symmetry. This gives the number of distinct configurations of a figure we can get while rotating, before we complete one full rotation, for which the rotational symmetry holds.

3 The Theorem

The theorem deals with the case where a polygon has rotational symmetry but the number of sides is a prime number. To get a proof idea, the following can be some questions one can ask about a polygon with rotational symmetry:

- What is the relationship between the number of sides and its order of rotational symmetry?
- What happens to order when the sides is prime?
- What condition must a polygon with a composite number of sides satisfy to have rotational symmetry?

3.1 How I got the proof idea

Any regular n -gon, has a rotational symmetry of order n . Some polygons have rotational symmetry even if they are not regular. The most basic example is a rectangle, which has a rotational symmetry of order 2 and it has four sides, which is not a prime number. The claim is that if the number of sides is prime, and it has a rotational symmetry, it has to be a regular polygon.

To move towards the proof, we first need to examine the nature of non-regular polygons with rotational symmetry. The simplest cases can be created by altering the regular polygons. Let us take, for example, a square. Since it has even number of sides, there is a unique opposite side for each side. If we choose a pair of opposite sides and stretch them uniformly, we will get a polygon with a rotational symmetry of order two. We can carry out the same process to get non regular polygons with even number of sides with rotational symmetry.

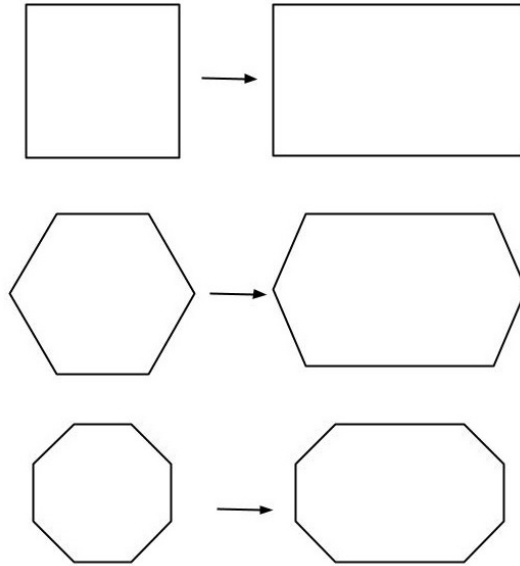


Figure 1: Generating even sided polygons with rotational symmetry

Now let us see how to generate such polygons if the number of sides is odd composite. The smallest of such numbers is nine. We can achieve it by morphing a regular nonagon. If we take a regular nonagon, choose one side, and stretch it, it loses the rotational symmetry. Let $a_1, a_2, a_3, \dots, a_8, a_9$ be the sides of the nonagon. Without loss of generality, let's stretch a_1 . Then, if we want a rotational symmetry to retain, we need to stretch some other sides uniformly.

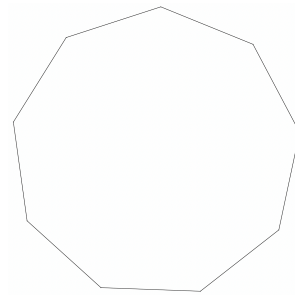


Figure 2: A regular nonagon

When we think of the ways of stretching other sides, we realise that only if we stretch two more sides periodically, will the nonagon have a rotational symmetry of order three. That means we need to stretch every third side uniformly after a_1 , that are a_4 and a_7 . Another nonagon with rotational symmetry can be generated by choosing another set of three periodic sides and stretching their lengths uniformly. Without loss of generality, let us assume that we stretch a_2, a_5 and a_8 uniformly. Then, the nonagon will have sides of three different lengths. Also, let every three consecutive sides be unequal. Let us call three consecutive sides as one **period**. Then,

Three periods: $\{a_1, a_2, a_3\}$, $\{a_4, a_5, a_6\}$, $\{a_7, a_8, a_9\}$

Here,

$$a_1 \neq a_2 \neq a_3,$$

$$a_4 \neq a_5 \neq a_6,$$

$$a_7 \neq a_8 \neq a_9$$

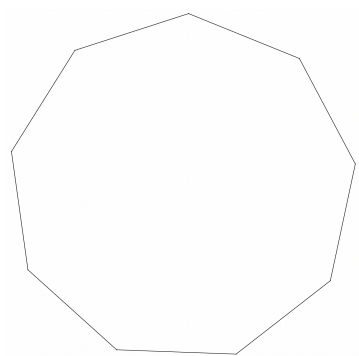
but,

$$a_1 = a_4 = a_7$$

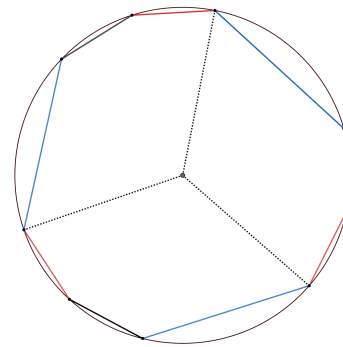
$$a_2 = a_5 = a_8$$

$$a_3 = a_6 = a_9$$

Let us call these equal sides as **periodic sides**.



(a) Regular nonagon



(b) An *irregular* nonagon with rotational symmetry

Figure 3: Rotational symmetry in nonagon

Definition 3.1. In any polygon with rotational symmetry with order > 1 , sides j_1 and j_2 are said to be periodic if there exists a natural number k such that after rotating j_1 around the centre of rotation at an angle $k\theta$, it coincides with j_2 . The set of all sides that a given side coincides with under such rotations, is called a set of periodic sides.

If the order of symmetry is s , when rotated, any side j has to coincide with exactly s sides, including itself. So, there should be s sides of equal lengths as j . Also, to satisfy rotational symmetry, these equal sides have to be placed uniformly at an angle equal to smallest angle of rotation, that is, $\frac{360^\circ}{s}$ away from their next equal side. So the sides with equal length as j should be placed uniformly. So, among n sides and s slots, j should be repeated every $\frac{n}{s}$ sides. Hence, periodicity of sides with equal length as j

should be $\frac{n}{s}$. Let $\frac{n}{s} = p$, and then the set of any p consecutive sides is called a **period**. Hence, we have that,

$$\text{number of sides} = \text{order of rotational symmetry} \times \text{number of sides in each period}.$$

3.2 Order = Number of periods

We were examining the nature of rotational symmetry in non-regular polygons. Now it seems that we have reached a conclusion. In polygons with even number of sides, there were two periods and each period had $\frac{n}{2}$ sides. In the nonagon, there were three periods and each period had $\frac{n}{3}$ sides. This means, we can generate irregular polygons with rotational symmetry using the factors of their number of sides.

For example, we can generate an irregular 12-gon, apart from the one generated by stretching only the opposite sides, which has an order 2. We can do this by,

1. choosing a regular 12-gon.
2. choosing of four periods of three consecutive sides.
3. stretching the respective periodic sides uniformly.

Here, every fourth side is equal. Since each side takes four identical positions in each period as we rotate, the order of symmetry is four. Similar process can be done to obtain a polygon of order of rotational symmetry three. Indeed, there is no other way than creating periodic equal sides.

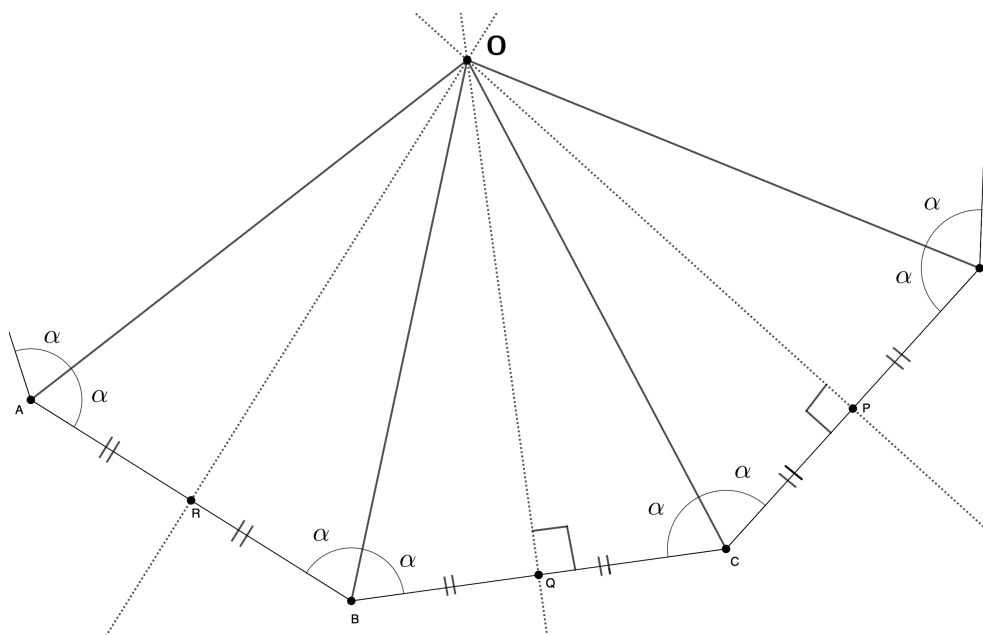
We will be using the standard notation for number of periods as s and number of sides in each period as p .

Now, an analogous argument can be made for vertices as well. Each vertex will coincide with its next $\left(\frac{n}{s}\right)^{\text{th}}$ vertex. We call the set of such vertices as periodic vertices. As the order is s , there have to be s vertices in each set of periodic vertices. These periodic vertices must lie on a circle with centre same as the centre of rotation. Because when they are rotated, they follow the path of a circle, and unless the next periodic vertex is on that circle, the vertex cannot coincide with the next vertex in its period. Now we will use all these results to prove the main theorem. These circle of different sets of periodic vertices may have different radii length but they they are all centred on the centre of rotation of the polygon.

Theorem 3.2. *Any polygon with prime number of sides and rotational symmetry with order > 1 , is a regular polygon.*

Proof. Let there be an n – gon with rotational symmetry of order $s > 1$. If n is prime, then we will either have $p = 1$ or $s = 1$. We are not dealing with the case where $s = 1$. So, if $p = 1$, then $s = n$. So, there are n periods with one side in each period. Since the periodic sides have to be equal, all of n sides will be of equal length.

As $s = n$, all n vertices of the polygon are periodic. Hence, they all lie on a circle. Let A and B be any two adjacent vertices and O be the centre. When we join all vertices with the centre, we get n triangles. Then, all sides joining vertices will be the sides of the polygon, which we proved that they are equal. Also, in the sides OB and OA in any triangle will be equal because all OB and OA are the radii of the same circle. Hence by *SSS* criteria for congruent triangles, we get that all triangles of the form AOB are congruent to each other where A and B are vertices and O is the centre of rotation. Due to corresponding equal angles of these triangles, the complete angle at the centre will be divided into n equal parts, each measuring $\frac{360^\circ}{n}$. Hence, $\angle AOB = \frac{360^\circ}{n}$ for any adjacent vertices A and B .



Let $\triangle A_1OB$ and $\triangle BOA_2$ be two such congruent triangles with OB as a common side. By corresponding parts, $\angle OBA_1 = \angle OBA_2 = \angle OA_2B = \angle OA_1B$, let them be equal to α . This α is a constant because all the triangles are congruent.

By angle sum property of a triangle in $\triangle BOA_2$, $\alpha + \alpha + \frac{360^\circ}{n} = 180^\circ$. Then, $2\alpha = \left(\frac{180(n-2)}{n}\right)^\circ$. See that 2α is a constant in an n -gon. All the internal angles of this polygon will be of the form A_1BA_2 , which is equal to 2α . Hence the polygon is equiangular. $\square$

Acknowledgements

The author would like to thank Swati Sircar of Azim Premji University for providing support and guidance. The author would also like to thank Prof. Mohan R of Azim Premji University for providing guidance in editing and reviewing.



Rajkumar Shirol is a BSc BEd graduate from Azim Premji University, with a strong interest in mathematics education. He plans to pursue a master's degree and teach mathematics at the secondary school level. He hopes to contribute to making quality mathematics education accessible to students across the country. Outside academics, he enjoys motorcycling, photography, and video editing. He looks forward to feedback on this article—please feel free to share your thoughts via email.

4 A Crazy Infinite Series Paradox

Sai Dibyanshu

Email: saidibyanshubehera@gmail.com

Introduction

Infinite series have fascinated mathematicians for long as they can be used to represent and study a wide variety of functions. They have applications in various fields such as physics and engineering. Particularly interesting series include geometric series and the harmonic series. Their study has led to insights into convergence, divergence, and the nature of infinity.

In this paper, we use algebraic manipulations and number theory to uncover some paradoxes connected with infinite series. They reveal the counter-intuitive and often baffling nature of Infinity.

This paper is merely an attempt to share some ‘cool, crazy’ aspects of modern mathematics.

I threw a single pebble, watching it create a ripple across the surface, but the depth of the ocean remains undiscovered, vast, and rich.

A formula from number theory

We shall make use of a well-known formula for the sum of the factors of a number. Given any positive integer n , let its prime factorization be

$$n = p^a \times q^b \times r^c \times \cdots, \tag{1}$$

where $p, q, r, \dots$ are prime numbers and $a, b, c, \dots$ are positive integers. Then, to find the sum of all the factors of n , we only need to expand the product

$$(1 + p + p^2 + \dots + p^a) \times (1 + q + q^2 + \dots + q^b) \times (1 + r + r^2 + \dots + r^c) \times \dots \quad (2)$$

This gives the sum of all the factors of n . To see why, observe that when we expand the product fully, the terms give the different factors of n , with no factor missed out. We shall make frequent use of this simple result.

The first paradox

For any positive integer N , consider the quantity S defined thus,

$$S = \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^N} \right) \times \prod_{p \in \mathbb{P}, p \geq 3} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots + \frac{1}{p^N} \right), \quad (3)$$

where $\mathbb{P}$ is the set of odd primes, $\{3, 5, 7, 11, 13, 17, \dots\}$.

Observe what happens as $N \rightarrow \infty$. Every term of the form $\frac{1}{2^n}$ where $n \in \mathbb{N}$ is counted precisely once each. So we may write

$$S = \sum_{n \in 2\mathbb{N}} \frac{1}{n}. \quad (4)$$

However, we also note the following:

$$\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^N} = 1 - \frac{1}{2^N} \rightarrow 1 \quad \text{as } N \rightarrow \infty. \quad (5)$$

This implies that we may write the expression for S as follows,

$$S = \prod_{p \in \mathbb{P}, p \geq 3} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots + \frac{1}{p^N} \right). \quad (6)$$

Observe what happens as $N \rightarrow \infty$. Every term of the form $\frac{1}{n}$ where $n \in 2\mathbb{N} - 1$ is counted precisely once each in the above infinite product. So we may write

$$S = \sum_{n \in 2\mathbb{N} - 1} \frac{1}{n}. \quad (7)$$

In other words, the sum of the reciprocals of the even numbers $2, 4, 6, 8, 10, \dots$ is equal to the sum of the reciprocals of the odd numbers $1, 3, 5, 7, 9, \dots$

This seems very counter-intuitive. But such is the nature of these paradoxes.

Further experimentation

We now make use of the well-known technique of telescoping to uncover another paradoxical result. We start by looking at a frequently seen problem of a series summation: to find X where

$$X = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots \quad (\text{to infinity}). \quad (8)$$

Simple partial fraction decomposition and manipulation leads us to:

$$\begin{aligned} X &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots \\ &= 1, \end{aligned} \quad (9)$$

after mass cancellation. So this infinite series converges to 1. (This result is well-known.)

To uncover the paradox, in the expression (8) for X we perform another set of manipulations: we apply the partial fraction decomposition only to the first fraction, the third fraction, the fifth fraction, and so on. Here is what we get.

$$X = \frac{1}{1} - \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4 \times 5} + \frac{1}{5} - \frac{1}{6} + \cdots \quad (\text{to infinity}). \quad (10)$$

In the above we incorporate the result obtained earlier about the sum of the reciprocals of the even numbers and the sum of the reciprocals of the odd numbers. A mass cancellation results, and we obtain the following:

$$\frac{1}{2 \times 3} + \frac{1}{4 \times 5} + \frac{1}{6 \times 7} + \cdots = 1. \quad (11)$$

If we compare this result with (9), it is contradictory.

Concluding remarks

The above paradoxes show that we cannot treat infinite series as simple algebraic objects and perform on them the manipulations we use to solve algebraic equations and simplify algebraic expressions. Infinity has to be handled in a somewhat different way. Otherwise we reach paradoxical and contradictory conclusions.

Let us consider (3) again but with some modification. Let

$$A = \prod_{p \in \mathbb{P}, 3 \leq p \leq M} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^N} \right) \quad (12)$$

and

$$B = \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^N} \right) \times A = \left(1 - \frac{1}{2^N} \right) A, \quad (13)$$

where M, N are large positive integers. Then it is perfectly valid to say that A is the sum of the reciprocals of some finite (and large) subset of the odd numbers, and B is the sum of the reciprocals of some finite (and large) subset of the even numbers. But the conclusion that $A = B$ as $M, N \rightarrow \infty$ is not meaningful, because both these quantities become infinitely large as $M, N \rightarrow \infty$. So, we cannot take this step.

Similarly, in the second paradox, which is of the form $\infty = \infty + k$, we cannot do a simple cancellation as we do when we are simplifying algebraic expressions and conclude that $k = 0$. We cannot treat infinity in the same way that we treat ‘ordinary’ numbers! So, we cannot take this step either.

What all this tells us is that we need a different set of rules when we are dealing with infinity.

Sai Dibyanshu is a student at Mother’s Public School, Bhubaneswar, studying in Grade X.

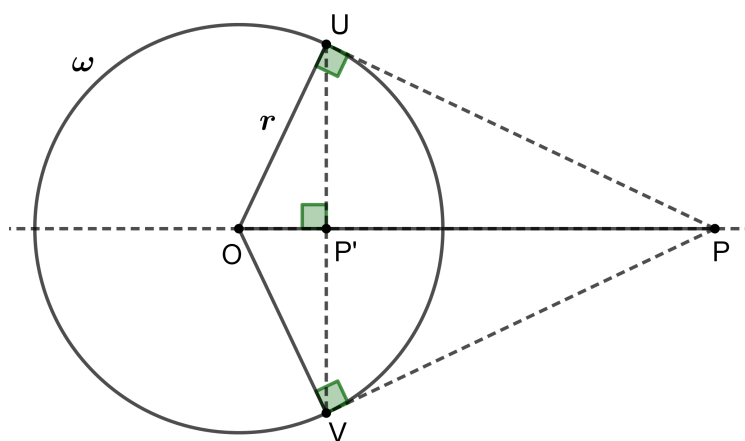
5 The Nine-point Circle of a Triangle – Part 3

Shailesh Shirali

The Valley School KFI,
Thatguni Post, Kanakapura Road, Bengaluru 560082, India
Email: shailesh.shirali@gmail.com

In Part 1 of this article we introduced the Euler line and the nine-point circle of a triangle, and in Part 2 we proved an astonishing tangency property of the nine-point circle that was first discovered by Karl Feuerbach: that the nine-point circle of a triangle is internally tangent to the incircle of the triangle and externally tangent to the three ex-circles. The proof made heavy use of distance computations. Now we prove the same result using a geometrical operation known as inversion.

The inversion mapping



Given a fixed circle ω with radius r and centre O , we define the inverse $f(P)$ of a point $P \neq O$ to be the unique point P' on ray OP such that

$$OP' \cdot OP = r^2.$$

The map f does not apply to O itself; so, the domain of f is $\mathbb{R} \setminus \{O\}$.

Figure 1: Definition of the inversion map f

A geometric procedure for locating the inverse point $f(P)$ for any given P is shown in Figure 1: (a) if P lies outside ω , draw tangents PU , PV from P to ω ; then $f(P)$ is the point P' where UV intersects OP ; (b) if P lies within ω , draw $UV \perp OP$ through P , with $U, V \in \omega$; then $f(P)$ is the point P' where the tangents to ω at U, V meet.

Remarks on the history of inversion

In [3] we read: “Inversion seems to have been discovered by a number of people contemporaneously, including Steiner (1824) ...and Kelvin (1845).” The latter (Lord Kelvin) found a way of using inversion to solve problems in electrostatics.

Features of the inversion map

Throughout, f refers to inversion in a fixed circle ω with centre O . Here are its main features.

Inv-1. The inverse $f(P')$ of P' is the original point P ; so f is its own inverse.

Inv-2. Points on the inverting circle ω map to themselves.

Inv-3. The inverse of a circle ω' concentric with ω is another circle ω'' concentric with ω .

Inv-4. (a) The inverse of a line ℓ through O (with the point O omitted) is ℓ itself (with the point O omitted): $f(\ell \setminus \{O\}) = \ell \setminus \{O\}$.

(b) Given a line ℓ not through O , the inverse of ℓ is a circle Γ through O (with the point O omitted). The centre of Γ lies on the perpendicular from O to ℓ .

Inv-5. a) Given a circle Γ through O , the inverse of Γ (with the point O omitted) is a line ℓ that is perpendicular to the diameter of Γ through O .

b) The inverse of a pair of intersecting circles that have common points O and P is a pair of intersecting lines through the inverse point $P' = f(P)$.

c) The inverse of a pair of tangent circles that touch at O is a pair of parallel lines.

Inv-6. The inverse of a circle Γ not passing through O is a circle Γ' not passing through O . In the particular case when Γ is orthogonal to the inverting circle ω , the circles Γ' and Γ coincide.

Inv-7. If two circles intersect at an angle θ , their inverses intersect at the same angle θ . In particular, orthogonal circles invert into orthogonal circles.

Inv-8. Under inversion in a circle ω with centre O and radius r , let the images of points A, B be A', B' . Then we have the following relation connecting the distances $A'B'$ and AB :

$$A'B' = \frac{r^2 \cdot AB}{OA \cdot OB}. \quad (1)$$

For the proofs of these propositions, we refer you to [1] and [2]. We now consider the tangency property of the nine-point circle.

Theorem. *The nine-point circle of a triangle is tangent to the incircle and to the three ex-circles.*

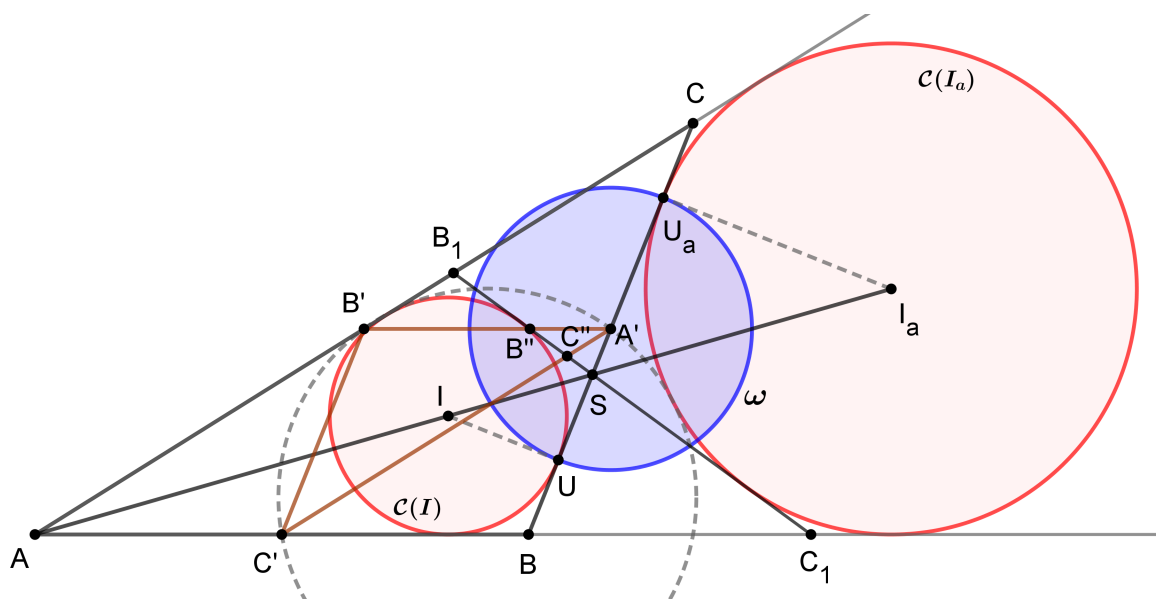


Figure 2: Proof of the tangency property of the nine-point circle: A', B', C' are the side midpoints; I is the incentre; I_a is the ex-centre opposite A

Proof

Let ABC be the given triangle (Figure 2). Its nine-point circle (shown dashed) passes through the side midpoints A', B', C' . We will show that it is tangent to the incircle,

$\mathcal{C}(I)$, and the ex-circle opposite vertex A , $\mathcal{C}(I_a)$. By symmetry tangency to the other two ex-circles will follow.

Let $\mathcal{C}(I)$ and $\mathcal{C}(I_a)$ touch BC at U and U_a respectively. Note that U and U_a are symmetrically placed on BC . Let B_1, C_1 be the images of B, C under reflection in AI_a . Then B_1C_1 is tangent to both $\mathcal{C}(I)$ and $\mathcal{C}(I_a)$. Let B'', C'' be the points where B_1C_1 intersects $A'B', A'C'$ respectively.

Let $\omega = \mathcal{C}(A')$ be the circle on UU_a as diameter. Since IU and I_aU_a are tangent to ω , it follows that $\mathcal{C}(I)$ and $\mathcal{C}(I_a)$ are orthogonal to ω . [It follows that under inversion in \$\omega\$, \$\mathcal{C}\(I\)\$ and \$\mathcal{C}\(I_a\)\$ invert into themselves.](#)

Observe that the nine-point circle ($A'B'C'$) passes through A' , which is the centre of the inverting circle ω . Therefore, the nine-point circle inverts into a line. [If we can show that this line is \$B_1C_1\$, then the desired conclusion will immediately follow.](#) To show this, it suffices to show that points B', C' invert into the points B'', C'' respectively.

So, we must show that $A'B'' \cdot A'B' = A'U^2$. It will then follow, by symmetry, that $A'C'' \cdot A'C' = A'U^2$, and our task will be done. Let us compute the relevant lengths and check whether the stated equality holds.

Using the standard symbols (a, b, c for the sides, s for the semi-perimeter), we have,

$$BU = s - b = CU_a, \quad \therefore UU_a = a - 2(s - b) = b - c, \quad A'U = \frac{b - c}{2}. \quad (2)$$

Note: We have drawn Figure 2 assuming that $b > c$. Consequently, points B, U, S, A', U_a, C occur in the order shown on BC . If, instead, we have $c > b$, then these six points would occur in the order B, U_a, A', S, U, C , and we would have $A'U = \frac{1}{2}(c - b)$. In the analysis shown below, we continue to assume that $b > c$.

The angle bisector theorem tells us that

$$BS = \frac{ac}{b + c}, \quad CS = \frac{ab}{b + c}, \quad \therefore A'S = \frac{1}{2} \left(\frac{ab}{b + c} - \frac{ac}{b + c} \right) = \frac{a(b - c)}{2(b + c)}. \quad (3)$$

Also,

$$BC_1 = AC_1 - AB = AC - AB = b - c, \quad B_1C = b - c. \quad (4)$$

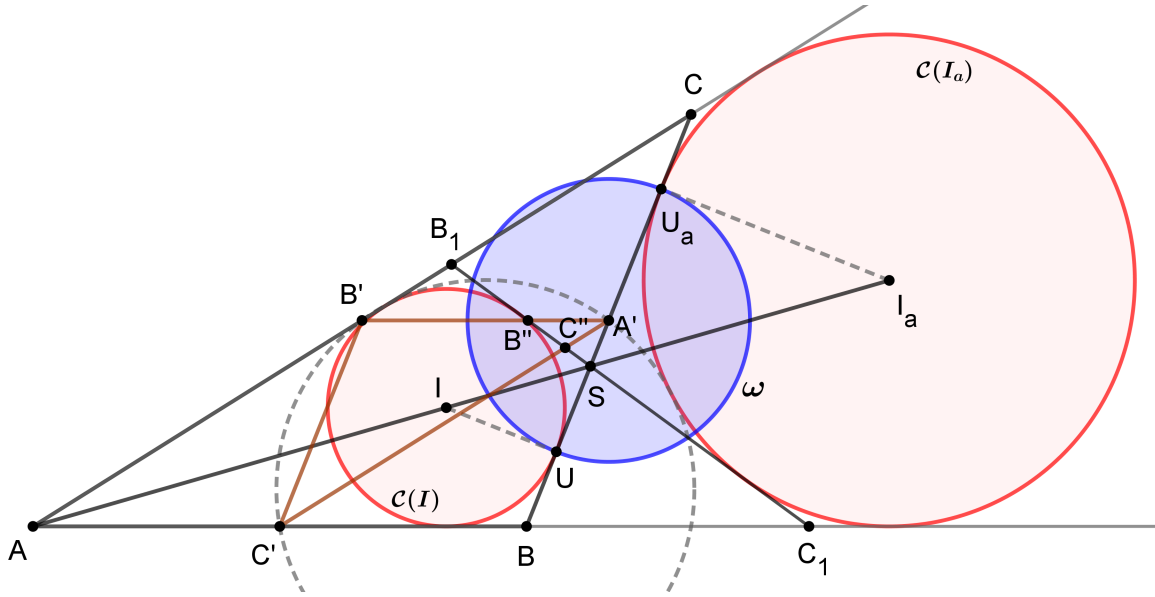


Figure 3: The same figure repeated for convenience

Next, since $A'B' \parallel BA$ and $A'C' \parallel CA$, we have,

$$\triangle SA'B'' \sim \triangle SBC_1, \quad \triangle SA'C'' \sim \triangle SCB_1, \quad (5)$$

hence,

$$\frac{A'B''}{BC_1} = \frac{A'S}{BS}, \quad \therefore A'B'' = (b-c) \cdot \frac{a(b-c)}{2(ac)} = \frac{(b-c)^2}{2c}, \quad (6)$$

implying that

$$A'B' \cdot A'B'' = \frac{c}{2} \cdot \frac{(b-c)^2}{2c} = \frac{(b-c)^2}{4} = A'U^2, \quad (7)$$

making use of (2).

In the same way we prove that $A'C' \cdot A'C'' = A'U^2$.

Invoking the ‘intersecting chords theorem’ (also called the ‘power chord theorem’) we deduce that B', C' invert into B'', C'' respectively, and hence that the nine-point circle inverts into line $B''C''$.

Since $B''C''$ is tangent to $C(I)$ and $C(I_a)$, it follows that the nine-point circle is tangent to $C(I)$ and $C(I_a)$, and, by symmetry, to the other two ex-circles too. $\square$

The reader will surely agree that this is an extremely elegant proof of the Feuerbach theorem. It is compact, but its compactness evidently draws from the properties (previously established) of the inversion map.

Remark

Very few college students would have an occasion today to study the above proof (or, for that matter, any of the other proofs of the theorem), and this seems a pity. Perhaps some space could be allotted for studying this proof when the extended plane is being studied (this is the ordinary plane extended by adjoining the ‘point at infinity’), and a connection is made with stereographic projection (which is a map from the surface of a sphere to the extended plane, using rays emanating from the North Pole of the sphere; here, the North Pole itself projects to the point at infinity).

Application: The Peaucellier–Lipkin linkage

Before closing, we describe a remarkable application of inversion to an engineering problem: that of transforming rotary motion into perfect straight-line motion, and vice versa, using a linkage mechanism: the *Peaucellier–Lipkin linkage*, named after its inventors, Charles-Nicolas Peaucellier and Yom Tov Lipman Lipkin (see [8] for details).

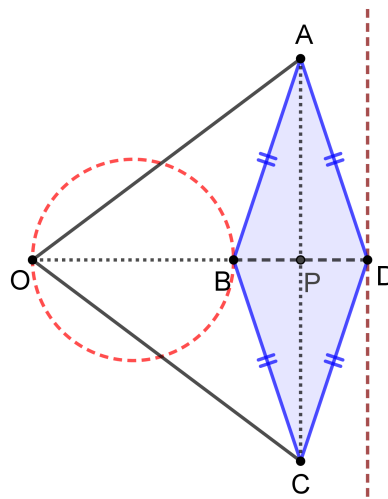


Figure 4: The Peaucellier–Lipkin linkage: $OA = OC$, and $AB = BC = CD = DA$. Segments AC and BD have a common midpoint, P .

The device is depicted schematically in Figure 4. It consists of six bars OA , OB , AB , BC , CD , DA , linked as shown. Bars OA , OC have equal length, and bars AB , BC , CD , DA have equal length too (so OCA forms an isosceles triangle, and $ABCD$ forms a rhombus). We show below that **under inversion in a circle with centre O and suitable radius, points B and D invert into one another**. Hence, if B describes a circle passing through O (the red, dashed circle), then D will describe a straight line (the brown,

dashed line). We will thus have achieved the objective of transforming rotary motion into perfect straight-line motion.

To show that under inversion in a circle with centre O and suitable radius, points B and D invert into one another, we first note that points O, B, D are collinear. This follows because O, B, D are equidistant from A and C and thus lie on the perpendicular bisector of AC .

To show the inversion property, it suffices to show that $OB \times OD$ is a constant, fixed by the lengths of the bars of the mechanism. We now have,

$$\begin{aligned} OB \times OD &= (OP - BP) \times (OP + PD) = OP^2 - BP^2 \quad (\text{since } BP = PD) \\ &= OA^2 - AP^2 - BP^2 = OA^2 - (AP^2 + BP^2) = OA^2 - AB^2 = \text{constant}. \end{aligned}$$

It follows that point B and D invert into one another, as claimed. We have achieved what we set out to do — conversion of circular motion into perfect straight line motion! $\square$

For a YouTube animation of the mechanism, see [11].

Bibliography

- [1] H S M Coxeter and S L Greitzer, *Geometry Revisited*. Mathematical Association of America. Chapter 5, “Inversive Geometry”
- [2] Dan Pedoe, *Geometry, a comprehensive course*. Cambridge University Press (1970). Chapter 2, “Circles”
- [3] Wikipedia, “Inversive geometry.” From https://en.wikipedia.org/wiki/Inversive_geometry
- [4] Alexander Bogomolny, “Inversion: Reflection in a Circle: What is it?” From <https://www.cut-the-knot.org/Curriculum/Geometry/SymmetryInCircle.shtml>
- [5] Wikipedia, “Feuerbach point.” From https://en.wikipedia.org/wiki/Feuerbach_point
- [6] Weisstein, Eric W. “Feuerbach’s Theorem.” From *MathWorld*—A Wolfram Web Resource. <https://mathworld.wolfram.com/FeuerbachsTheorem.html>
- [7] Wikipedia, “Nine-point circle.” From https://en.wikipedia.org/wiki/Nine-point_circle
- [8] Wikipedia, “Peaucellier–Lipkin linkage.” From https://en.wikipedia.org/wiki/Peaucellier-Lipkin_linkage
- [9] Weisstein, Eric W. “Nine-Point Circle.” From *MathWorld*—A Wolfram Web Resource. <https://mathworld.wolfram.com/Nine-PointCircle.html>

- [10] Weisstein, Eric W. “Inversion.” From MathWorld—A Wolfram Web Resource.
<https://mathworld.wolfram.com/Inversion.html>
- [11] MekanikaEngineerikaTV, “Motion animation using Matlab: Peaucellier–Lipkin linkage (straight-line mechanism).” From
https://www.youtube.com/watch?v=9H_s1Y3amsM



Shailesh Shirali, Head of the Teacher Education Programme KFI, and based at The Valley School, Bangalore, has been with KFI schools since the early 1980s. Formerly Director of Sahyadri School KFI and Principal of Rishi Valley School, he has decades of experience in school level math education and has authored many high school math books. He is keenly interested in fostering inquiry into fundamental issues among high school teachers and students. He also has a deep interest in math exposition and the history of mathematics and ancient Indian mathematics.

6 Repeating Coprime Integers Eventually Ending at 1

Anand Prakash

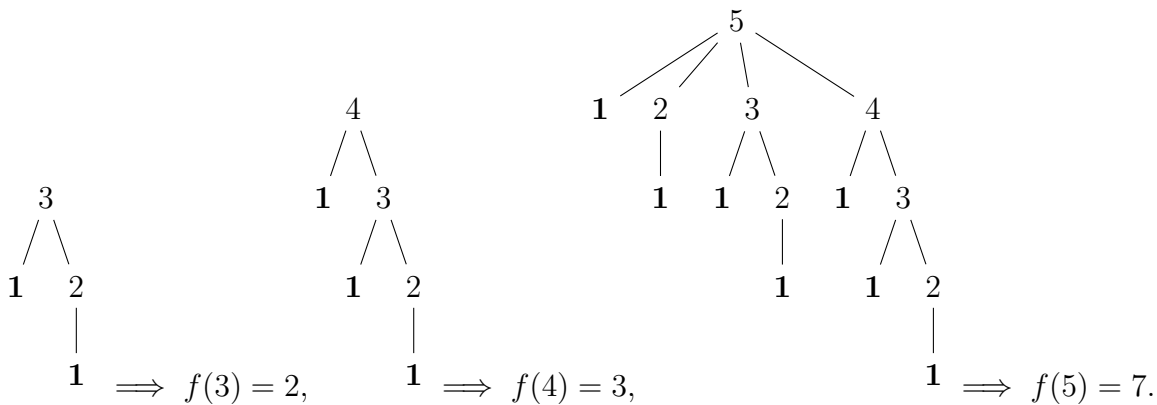
Email: prakashanand805@gmail.com

We define a function f on the set of natural numbers as follows. First, set

$$f(1) = f(2) = 1.$$

In general, for any $n > 1$, consider the set of positive integers $k < n$ that are coprime to n ; say $a_1, a_2, \dots, a_m$. For each $a_i > 1$ again write down the positive integers that are smaller than and coprime to a_i . In this manner, keep proceeding until we have a sequence of 1s. Now take $f(n)$ to be the total number of 1s.

Consider the process for $n = 3, 4, 5$:



We can see that $f(n)$ is recursively defined by

$$f(n) = f(a_1) + f(a_2) + \dots + f(a_m).$$

Here are a few more examples:

$$\begin{aligned}
 f(6) &= f(1) + f(5) = 8, \\
 f(7) &= f(1) + f(2) + f(3) + f(4) + f(5) + f(6) = 22, \\
 f(8) &= f(1) + f(3) + f(5) + f(7) = 32, \\
 f(9) &= f(1) + f(2) + f(4) + f(5) + f(7) + f(8) = 66, \\
 f(10) &= f(1) + f(3) + f(7) + f(9) = 91.
 \end{aligned}$$

The sequence $f(n)$ grows rather fast! For instance, $f(20) = 32741$.

Here are some more observations.

- Since $n - 1$ and n are coprime, we have $f(n - 1) < f(n)$ for $n \geq 3$. Hence $m < n$ implies $f(m) < f(n)$ for $n \geq 3$.
- If P_n is the n -th prime, we have

$$\begin{aligned}
 f(P_n) &= f(1) + \cdots + f(P_{n-1} - 1) + f(P_{n-1}) + f(P_{n-1} + 1) + \cdots + f(P_n - 1) \\
 &= 2f(P_{n-1}) + f(P_{n-1} + 1) + \cdots + f(P_n - 1).
 \end{aligned}$$

- Similarly, $f(P_{n-1}) - 2f(P_{n-2})$ is the sum of f -values of all numbers strictly between P_{n-2} and P_{n-1} . In this manner, we have a quicker way to calculate $f(P_n)$ recursively.

For example, as 8, 9, 10 are the composites between 7 and 11, we have

$$f(11) = 2f(7) + f(8) + f(9) + f(10) = 44 + 32 + 66 + 91 = 233.$$

Here is a table of further values of $f(n)$:

n	12	13	14	15	16	17	18	19	20
$f(n)$	263	729	1,038	2,059	3,119	7,674	8,666	24,014	32,741

It is difficult to find good, tight bounds for $f(n)$. Here is a somewhat lazy one.

Claim. *Let $N > 1$. Then $f(N) < 2^{N-1}$.*

We can prove this by strong induction. Note that $f(2) = 1 < 2^{2-1}$. Assume the inequality holds for all n up to $N - 1$. Then,

$$f(N) \leq \sum_{i=1}^{N-1} f(i) < \sum_{i=1}^{N-1} 2^{i-1} = \frac{2^{N-1} - 1}{2 - 1} < 2^{N-1}. \quad \square$$

We end by posing two problems for further exploration. Are there better bounds for $f(n)$? Also, are there other recursive formulae involving the function f ?



Anand Prakash runs a garment shop in Kesariya, a village in Bihar. He is fascinated by classical Hindi music as well as by the beauty of nature.

7 The Champagne Problem – Distilled from Hilbert 17

B. Sury

Indian Statistical Institute

Bangalore

Email: surybang@gmail.com

1 Some Interesting History

A famous theorem by Lagrange says that any positive integer a is expressible as a sum of four squares of integers. Positivity is a key concept implicitly used here. Thus, one could analyze questions of this nature over real numbers where there is a notion of positivity. For example, let us consider real polynomials possibly in more than one variable. We write $f \in \mathbb{R}[x_1, \dots, x_n]$ to mean that it is a polynomial in n variables with real coefficients. The analogue of the hypothesis that $a \geq 0$ in positive integers, would then be all f such that the function $f > 0$ for all values of $x_1, x_2, \dots, x_n \in \mathbb{R}$. We call such f as *positive semi-definite* (psd). The 21-year old Minkowski presenting his Inaugural Dissertation in July 1885 on quadratic forms made the bold conjecture that there must exist homogeneous, real, psd polynomials of any degree > 2 in $n > 2$ variables which are *not* sums of squares of homogeneous real polynomials. At the public defense of this Dissertation, it was the task of Hilbert to attack it but the defense ended with Hilbert declaring that he “was convinced by Minkowski’s exposition that already for $n = 3$ there may well be such remarkable forms, which are so stubborn as to remain positive without allowing themselves to submit to a representation as sums of squares of forms.”

In 1888, Hilbert proved Minkowski’s ‘conjecture’ giving examples where a psd real polynomial f cannot be written as a sum of squares of polynomials. He studied it further and considered the problem of representing any psd $f \in R[x_1, \dots, x_n]$ as sums of squares of rational functions (elements of $\mathbb{R}(x_1, \dots, x_n)$). In 1893, he proved that this does happen for $n = 2$ (this corresponds to the 3-variable homogeneous case of Minkowski’s

conjecture). In 1899, he proved the remarkable ‘result’:

Any segment of length $f(x_1, \dots, x_n)$ which can be constructed from given lengths $x_1, \dots, x_n$ by using a ruler and compass, can already be constructed without a compass, provided $f(y_1, \dots, y_n)$ is a totally real algebraic number for any $y_1, \dots, y_n \in \mathbb{Q}$.

His ‘proof’ of this result required the truth of the (at that time) unproved assertion that any psd rational function in $\mathbb{Q}(x_1, \dots, x_n)$ is a sum of squares of rational functions in $\mathbb{Q}(x_1, \dots, x_n)$. This was a further motivation for him to formulate in his famous 1900 address the 17th problem which is the question:

If $f \in \mathbb{R}[x_1, x_2, \dots, x_n]$ is psd, is it necessarily a sum of squares of rational functions in $\mathbb{R}(x_1, x_2, \dots, x_n)$?

Instead of $\mathbb{R}$, if we consider $\mathbb{C}$ - this is a field which does not have an order - then we see that *every* polynomial in $\mathbb{C}[x_1, x_2, \dots, x_n]$ is a sum of squares of rational functions. The 17th problem was solved by E. Artin in 1926 in the affirmative. Artin proved it as an existence theorem. His remarkable proof opened up the new subject of model theory and nowadays his argument is viewed as a special case of Tarski’s transfer principle. The proof also brought into the forefront the *real spectrum* of rings such as $\mathbb{R}[x_1, \dots, x_n]$ and started the subject of real algebraic geometry. In a series of articles, Professor Jugal Verma will outline the proof and its many applications. The proof works for more general ‘real-closed’ fields and such fields were studied by Artin and Schreier in a paper in the same volume where Artin’s solution of Hilbert’s 17th problem was published. Indeed, the Artin-Schreier paper is from page 85 to 99 and is followed by Artin’s paper from page 100 to 115. A key fact that is in the background but plays a significant role in all this is that a field in which -1 is not a sum of squares is precisely the sort of field which has an ordering.

We mention a few words about the connection of Artin-Schreier’s 1926 work with mathematical logic. Tarski proved later in 1948 that the theory of real-closed ordered fields admits ‘quantifier elimination’ in the language of ordered rings. That is, every ‘formula’ is equivalent to a quantifier-free formula. Thus, the theory of real-closed fields is ‘model-complete.’ In other words, if $E \subseteq F$ are real-closed fields, then an ‘elementary sentence’ about ordered fields with parameters in E , holds good in F if and only if it holds good in E . This is crystallized as Tarski’s transfer principle which asserts that every elementary sentence about ordered fields which holds in $\mathbb{R}$ also holds in every real-closed field.

The basic idea of Artin’s theorem can be described roughly as follows. Although $\mathbb{R}$ (or more generally, any ‘real-closed field’) has a unique notion of positive elements, the function field $\mathbb{R}(x_1, \dots, x_n)$ has several possible notions of positivity. If f is not a sum of squares in $\mathbb{R}(x_1, \dots, x_n)$, then there would be some ordering (equivalently, a concept of positive elements) on $\mathbb{R}(x_1, \dots, x_n)$, under which f would be negative. Then, there

would be a ‘specialisation’ $f(a_1, \dots, a_n)$ which would be negative. This last step is the key one and is a form of the Tarski transfer principle. Some properties of polynomials over the real field like the intermediate value theorem and others were proved by Sturm in the 1880’s and go through for all real-closed fields. These are used by Artin in his proof.

2 The Champagne Problem

Among problems arising from Hilbert’s 17th problem, one is the so-called *champagne problem*. In the early 1980s, E. Becker applied these abstract methods to deduce that the rational function $B(t) = \frac{1+t^2}{2+t^2}$ is a sum of $2n$ -th powers of elements of $\mathbb{Q}(t)$ for every $n \geq 1$. Due to the existential nature of the proof, Becker offered a bottle of champagne to the first person who could explicitly express $B(t)$ as a sum of $2n$ -th powers in $\mathbb{Q}(t)$. **This problem is not solved as yet!** However, it has been solved over the reals by Bruce Reznick in 1992; we will recall his answer below. We do not discuss Becker’s existential proof here. But, we first discuss briefly the background of the 17th problem itself.

3 Hilbert’s 17th problem for $n = 1$

The proof that a psd $f \in \mathbb{R}[x]$ is a sum of squares of polynomials (and not just rational functions), is really simple and we give it now.

We first notice that every real root (if at all it exists) occurs with even multiplicity. To see this write: $f(x) = (x - \alpha)^n g(x)$ where $\alpha \in \mathbb{R}$ and $g(\alpha) \neq 0$. So if n is odd, then $(x - \alpha)^n$ and hence f would change sign in the neighbourhood of α , which contradicts the fact that f is psd. Hence each real root occurs with even multiplicity and we may write:

$$f(x) = c \prod (x - \alpha_i)^{2n_i} \prod (x - \beta_i)^{n_i} \prod (x - \bar{\beta}_i)^{n_i}$$

where $\alpha_i \in \mathbb{R}$ and $\beta_i \notin \mathbb{R}, c \in \mathbb{R}$ Let

$$h(x) = \prod (x - \beta_i)^{n_i} \prod (x - \alpha_i)^{n_i}.$$

So, $f(x) = ch(x)h(\bar{x})$; Now, $h(x) = p(x) + iq(x)$ for $p, q \in \mathbb{R}$ So, $f(x) = c(p(x)^2 + q(x)^2) = (\sqrt{c}p(x))^2 + (\sqrt{c}q(x))^2$.

4 Necessity of rational functions

Hilbert was the first to give an existential proof of a psd polynomial that is not a sum of squares of real polynomials. But, here is an explicit example due to Motzkin showing that for general n , it is not sufficient to work with polynomials. Consider

$$f(x, y) = (x^2 + y^2 - 3)x^2y^2 + 1 \in \mathbb{R}[x, y].$$

Clearly f is psd, as $\frac{x^2+y^2+\frac{1}{x^2y^2}}{3} \geq 1$. Suppose, if possible, $f = f_1^2 + f_2^2 + \cdots + f_n^2$ with $f_i \in \mathbb{R}[x, y]$. For all $i \leq n$, $\deg f_i \leq 3$, as the total degree of f is 6. Since $f(x, 0) = f(0, y) = 1$, each $f_i(x, 0)$ and $f_i(0, y)$ is a constant. Thus $f_i(x, y) = a_i + xy(b_i + c_i x + d_i y) \forall 1 \leq i \leq n$. Now $f = \sum f_i^2$; so the coefficient of x^2y^2 in $\sum f_i^2$ is -3 . Hence $\sum b_i^2 = -3$, which is a contradiction.

Higher degree examples.

Consider $f(x) = x^4 + nx^2 + 1$. It is a fact that it is a sum of 4-th powers in $\mathbb{R}(x)$. On the other hand we claim that f is not a sum of squares of real polynomials if n is large. To see this, suppose, if possible, that $f = \sum_{i=1}^N (a_i + b_i x)^4$. Comparing the coefficients of x^2 we have

$$n = 6 \sum_{i=1}^N a_i^2 b_i^2.$$

Also, we have $1 = \sum_{i=1}^N a_i^4$ and $1 = \sum_{i=1}^N b_i^4$. Using the Cauchy-Schwarz inequality now, we see immediately that

$$n \leq 6 \sum_{i=1}^N a_i^4 \sum_{i=1}^N b_i^4 = 6.$$

5 Hilbert's identities – $\mathbb{R}$ to the rescue

Becker's proof of the fact that $\frac{1+t^2}{2+t^2}$ is a sum of $2n$ -th powers of rational functions in t over $\mathbb{Q}$, actually demonstrates more general identities of the form

$$(x_1^{2n} + x_2^{2n} + \cdots + x_k^{2n})^m = f_1^{2nm} + \cdots + f_r^{2nm}$$

for some rational functions $f_i \in \mathbb{Q}(x_1, \dots, x_k)$ - of course existentially, as mentioned. In the case when $n = 1$, Hilbert had proved such identities with f_i s being polynomials;

hence the above identities are also sometimes called Hilbert's identities. More precisely, Hilbert's identities are:

$$(x_1^2 + \cdots + x_k^2)^m = \sum_{i=1}^N a_i (b_{i1}x_1 + \cdots + b_{ik}x_k)^{2m}$$

where $N = \binom{2m+k-1}{k-1}$ and a_i, b_{ij} are rational. Even though the analogues of the above identities over $\mathbb{Q}$ are unknown in explicit form (as we mentioned while stating the champagne problem), the existence of such identities over $\mathbb{Q}$ were already proved by Hilbert.

6 Reznick gets the real champagne

Over $\mathbb{R}$, for some small powers, the Hilbert identities are known in explicit form. For example:

$$(x^2 + y^2)^3 = \frac{4}{5} \left(x^6 + \left(\frac{x+y}{\sqrt{2}} \right)^6 + y^6 + \left(\frac{y-x}{\sqrt{2}} \right)^6 \right).$$

Note that these are NOT over $\mathbb{Q}$. In the same vein, B. Reznick obtained the following identity, thereby answering the champagne problem at least over the reals.

$$(x^2 + y^2)^m = \frac{2^{2m}}{v \binom{2m}{m}} \sum_{j=0}^{v-1} \left(\cos(j\pi x/v) + \sin(j\pi y/v) \right)^{2m}.$$

Taking $m = n, v = n + 2, x = \sqrt{2}, y = t$, and simplifying, one gets Reznick's remarkable formula:

$$\frac{1+t^2}{2+t^2} = A(n) \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} \lambda_j \left(\frac{L_i(\sqrt{2}, t) L_j(\sqrt{2}, t)}{2+t^2} \right)^{2n}$$

where

$$A(n) = \frac{2^{4n-2}}{n(n+2)^2 \binom{2n}{n}^2},$$

$$\lambda_j = 3n - (n+1) \cos(2j\pi/(n+2)),$$

$$L_j(x, y) = x \cos(2j\pi/(n+2)) + y \sin(2j\pi/(n+2)).$$

Bibliography

- [1] David Hilbert, Über die Darstellung definiter Formen als Summe von Formenquadraten, *Math. Ann.* 32 (1888), pp. 342-350.

- [2] Victoria Powers, Hilbert's 17th Problem and the Champagne Problem Victoria Powers, *The American Mathematical Monthly*, December 1996, pp.879-887.
- [3] B. Reznick, *Some concrete aspects of Hilbert's 17th Problem*, Seminarie de Structures Algebriques Ordonnees, Vol. 56, Equipe de Logique Mathematique, Universite Paris VII, 1996.



Sury is at I.S.I. Bangalore, and likes to interact with students of all age groups. The present article refers to Hilbert's 17th problem which was resolved by Emil Artin and later led to the introduction of methods from mathematical logic. Sury introduces this article with:

When we try to find art in mathematics,
and wonder if art and science really mix,
when we discuss Hilbert's 17,
mathematical logic enters unseen,
unexpectedly following Artin's mathematics.

8 A Spiral Prism on a Regular Base

Jyotirmoy Sarkar

Indiana University Indianapolis
Department of Mathematical Sciences
402 N Blackford Street
Indianapolis, IN 46202-3216, USA
Email: jsarkar@iu.edu

Abstract

Suppose a 2D regular n -gon rises uniformly from bottom to top and also rotates uniformly counterclockwise by a total angle of $2\pi/n$. This spiral transformation generates a 3D spiral prism, in which the top and bottom n -gons are vertical projections of each other, just as they are for a right prism. From the union of a prism and its spiral cousin, we find (1) the areas of various surfaces, (2) the volumes of various parts, and (3) the lengths of various curves.

Celebrate Creativity

They say that my wife has a green thumb: Whatever she plants, it grows. But I know it is not magic. It grows because she bestows so much love and care! During the last snowy winter, she grew an indoor creeper from a cutting of an older plant. Readers, take a good look at Figure 1. I admired her wisdom, patience, and diligence that birthed this thriving plant. Feeling somewhat incompetent to contribute anything worthwhile to her creative effort, I wondered: “What can I do instead?”

Although my wife was oblivious to the vase in which she housed her creeper, I was mesmerized by it. I resolved to popularize the square-based spiral prism. As much as I enjoyed the lively plant of my wife’s creation, which any visitor to our home can also enjoy, I sincerely hope that she and all of you, dear readers, will take some pleasure in the fruit of my creation — this mathematics paper.



Figure 1: The new indoor creeper and the vase containing it

1 A Square-based Right Spiral Prism

To describe the vase in Figure 1, let us first define a square-based right prism.

When a 2×2 square $ABCD$ in a flat plane moves vertically up by any distance, it generates a right, squared-based prism. The final (topmost) position is a square $\bar{A}\bar{B}\bar{C}\bar{D}$, with $\bar{A}$ vertically above A , etc. In fact, every point in the square moves vertically up. In particular, the center O of the square $ABCD$ moves vertically upward, generating a line called the axis of the prism. In addition, the four boundary lines of the square generate the four lateral surfaces of the prism. The prism has volume $4H$ unit³, and each of the four lateral surfaces has area $2H$ unit².

Next, imagine that, as the square moves up at a uniform speed, it also rotates uniformly about its center O counterclockwise. By the time the total rotation is $\pi/2$, suppose that the square has moved up by H units. The 3D solid object generated by the moving and rotating square is called a square-based right spiral prism (henceforth, a spiral prism). Figure 2(a) shows the union of a right prism (green) and a spiral prism (red). Despite the rotation, the axes of the prism and the spiral prism are the same line! The topmost square $\bar{A}\bar{B}\bar{C}\bar{D}$, also denoted by $D'A'B'C'$, is vertically above the bottommost square $ABCD$. Specifically, the point $D' = \bar{A}$ is vertically above A , the point $A' = \bar{B}$ is vertically above B , etc.

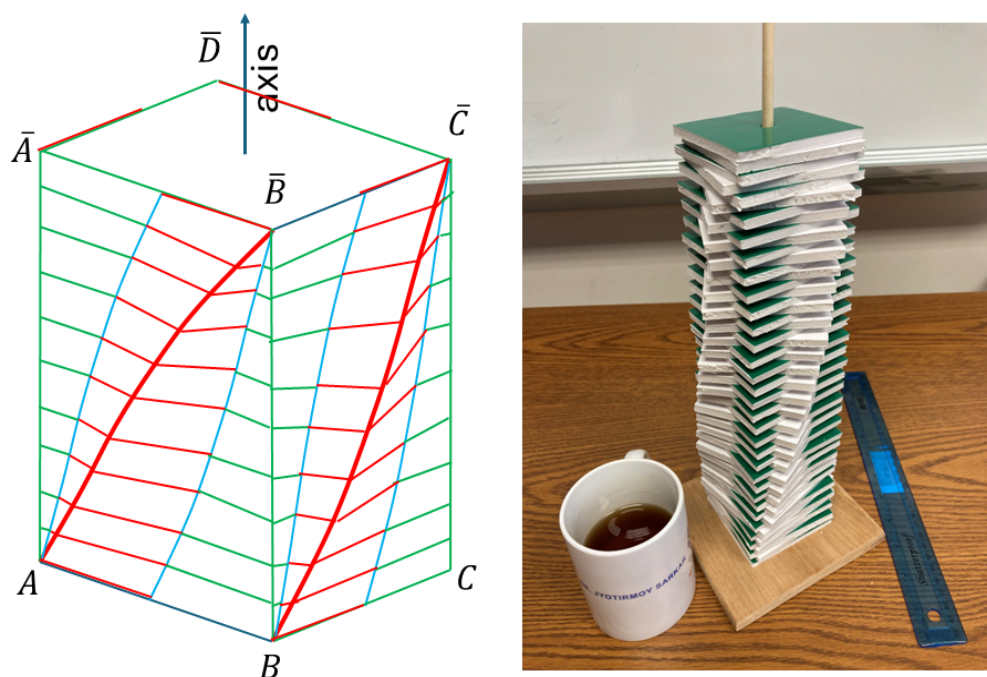


Figure 2: (a) The union of a prism (green) and a spiral prism (red) from the same square base, and (b) a physical model of a prism (green) and a spiral prism (white) from the same square base

Caution: Do not confuse this spiral prism with the so-called twisted rectangular prism. You can learn about those simpler objects from [1].

The volume of the spiral prism is the same as that of the original (regular) prism, both being $4H$ unit³. This claim follows from Cavalieri's principle, named after Bonaventura Francesco Cavalieri (1598-1647). It is stated below. See details in [2].

Cavalieri's principle: Suppose that two solid objects are situated between two parallel planes. If every plane parallel to these two planes intersects the two objects in cross-sections whose areas are in the same proportion, then the two objects have volumes in the same proportion. We give an example below.

Suppose that the axes of two long cylinders of equal radius r intersect orthogonally, forming a plane $\mathcal{P}$. Compare the solid of intersection $\mathcal{I}$ shown in Figure 3 with that of its inscribed sphere $\mathcal{S}$ (which you must imagine). Every plane parallel to $\mathcal{P}$ (hence, orthogonal to the line $\mathcal{L}$ joining the two vertices where four faces meet) intersects $\mathcal{I}$ and $\mathcal{S}$ producing cross-sections that are squares and inscribed circles, respectively, with areas in the ratio $4 : \pi$. So are their volumes. The volume of $\mathcal{S}$ being $(4/3)\pi r^3$, the volume of $\mathcal{I}$ is $(4/\pi)(4/3)\pi r^3 = (16/3)r^3$, free of π !

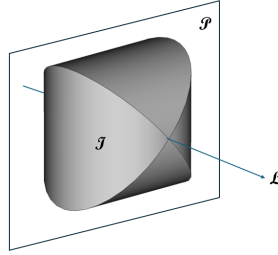


Figure 3: The orthogonal intersection of two identical cylinders

Sometimes we can compute the lateral surface area of a 3D object by comparing it with another object. In the above example, $\mathcal{L}$ passes through the centers of gravity of the cross-sections of $\mathcal{J}$ and $\mathcal{S}$, and the cross-sections are orthogonal to $\mathcal{L}$ and have perimeters in the ratio $4 : \pi$. Hence, the surface areas of $\mathcal{J}$ and $\mathcal{S}$ are in the same ratio $4 : \pi$. The surface area of $\mathcal{S}$ being $4\pi r^2$, the surface area of $\mathcal{J}$ is $(4/\pi)4\pi r^2 = 16r^2$, again free of π ! See details in [3].

Returning to the spiral prism and the right prism, can you now justify that their four lateral surfaces have the same total area $8H$ unit²? Imitate the justification in the previous paragraph. Traveling along the lateral surface of the spiral prism, what is the length of the locus $\mathcal{L}_A$ traced by the point A until it reaches $A' = \bar{B}$? Or, what is the length of the locus $\mathcal{L}_X$ traced by any other point X on AB until it reaches X' ? Also, what is the length of the vertical curve from B to $\bar{B} = A'$? We will answer these questions, but leave open the question of how to find the shortest path between any two points P and Q on the lateral surface of the spiral prism.

To facilitate answering the questions posed above, imagine that a red spiral prism (with a total rotation of $\pi/2$) is superimposed on a green right prism with an equal base square and equal height and with their axes coincident. In Figure 2(a), focusing on the green part and eliminating the red part, we see the prism. Likewise, focusing on the red part and eliminating the green part, we see the spiral prism. The part common to both the prism and the spiral prism is their intersection.

Figure 2(b) shows a discretized model of the right prism (green) and the spiral prism (white). By filling in the empty spaces (that is, by taking the convex hull only in the vertical direction), we see the union of the prism and the spiral prism. By discarding all the green and white right-triangular extensions (with empty spaces above and below) from each square in the model, we see the intersection of the prism and the spiral prism. By filling in the empty spaces between the green squares and discarding the right-triangular extensions of the white squares, we recover the prism. By filling in the empty spaces between the white squares and discarding the right-triangular extensions of the green squares, we construct the spiral prism.

We study the surface areas, the volumes of various parts, and the lengths of various curves of the prism, the spiral prism, their union, and intersection.

2 Surface Areas

Consider the union of the prism and the spiral prism and focus on the front face of the prism. This face is partially visible and partially covered by the spiral prism. See Figure 4(a).

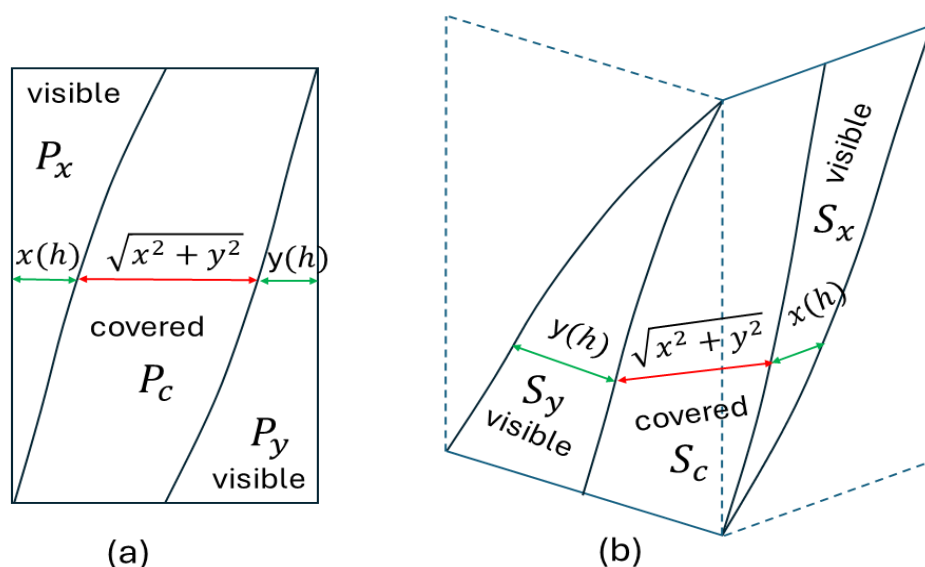


Figure 4: A lateral face (a) of the prism, and (b) of the spiral prism, seen from their union

A curve is formed where the spiral prism exits from the plane face of the prism. Let $x(h)$ denote the orthogonal distance of this curve from the left vertical boundary line of the front face of the prism. Another curve is traced where the spiral prism recedes into the plane face of the prism. Let $y(h)$ denote the orthogonal distance of this second curve from the right vertical boundary line.

To find the expressions for $x(h)$ and $y(h)$, we depict in Figure 5 the cross-section of the union (of the prism and the spiral prism) at height h . The cross-section consists of two overlapping squares rotated by an angle θ , which is related to h as follows: Due to simultaneous uniform vertical translation and uniform rotation, when the base square rotates by an angle $\theta \in [0, \pi/2]$, it moves up by a fraction $\theta/(\pi/2)$ of H ; that is, $h = (2H/\pi)\theta \in [0, H]$ units. Consequently, $\theta = (\pi/2H)h$.

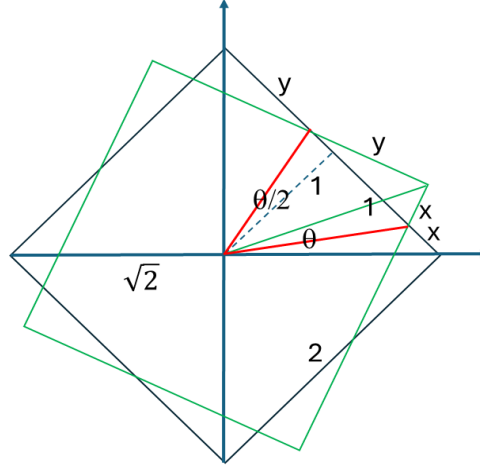


Figure 5: Two 2×2 concentric squares rotated by an angle θ

In Figure 4(a), we see that $x(0) = 0, y(0) = 1, x(H) = 1, y(H) = 0$. Also, referring to Figure 5, for any $h \in (0, H)$, writing $t = \tan(\theta/2) = \tan((h/H)\pi/4)$, we have

$$x(h) = 1 - \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = 1 - \frac{1-t}{1+t} = \frac{2t}{1+t}, \quad (1)$$

using trigonometric identity for the tangent of the difference of two angles; and

$$y(h) = 1 - \tan\left(\frac{\theta}{2}\right) = 1 - t. \quad (2)$$

From (1) and (2), it follows that

$$x^2(h) + y^2(h) = \left(\frac{2t}{1+t}\right)^2 + (1-t)^2 = \frac{(2t)^2 + (1-t^2)^2}{(1+t)^2} = \frac{(1+t^2)^2}{(1+t)^2},$$

whence, for all $h \in [0, H]$, we have

$$x(h) + \sqrt{x^2(h) + y^2(h)} + y(h) = \frac{2t}{(1+t)} + \frac{(1+t^2)}{(1+t)} + (1-t) = 2.$$

Thus, the entire width 2 of each face of the prism is split into three components by the spiral prism: a visible left component, a middle component covered by the spiral prism, and another visible right component. What are their areas?

Since $t = \tan(\theta/2) = \tan(h\pi/(4H))$, we have $dt = \sec^2(\theta/2) (\pi/(4H)) dh$, from which

$$dh = \frac{4H}{\pi} \frac{dt}{1+t^2}.$$

The area of the visible plane region on the front face of the prism to the left of $x(h)$, shown in Figure 4(a), and explained below, is

$$\begin{aligned}
 P_x &= \int_0^H x(h) dh \\
 &= \int_0^1 \frac{2t}{1+t} \frac{4H}{\pi} \frac{dt}{1+t^2} \\
 &= \frac{4H}{\pi} \int_0^1 \left\{ \frac{1}{1+t^2} + \frac{t}{1+t^2} - \frac{1}{1+t} \right\} dt \\
 &= \frac{4H}{\pi} \left\{ \frac{\pi}{4} + \frac{1}{2} \ln 2 - \ln 2 \right\} = H (1 - 2 \ln 2 / \pi).
 \end{aligned} \tag{3}$$

In the last line of (3), we used these three indefinite integrals: (1) $\int (1+t^2)^{-1} dt = \arctan(x)$, (2) $\int 2t(1+t^2)^{-1} dt = \ln |1+t^2|$, and (3) $\int (1+t)^{-1} dt = \ln |1+t|$.

Likewise, the area of the visible plane region on the front face of prism to the right of $y(h)$, also shown in Figure 4(a), and using the same integrals as above, is

$$\begin{aligned}
 P_y &= \int_0^H y(h) dh \\
 &= \int_0^1 (1-t) \frac{4H}{\pi} \frac{dt}{1+t^2} \\
 &= \frac{4H}{\pi} \int_0^1 \left\{ \frac{1}{1+t^2} - \frac{t}{1+t^2} \right\} dt \\
 &= \frac{4H}{\pi} \left\{ \frac{\pi}{4} - \frac{1}{2} \ln 2 \right\} = H (1 - 2 \ln 2 / \pi).
 \end{aligned} \tag{4}$$

It should not come as a surprise that $P_x = P_y$: After all, the front face of the prism in the union must exhibit a 180° rotation symmetry; that is, $x(h) = y(H-h)$, for all $h \in [0, H]$. This equality is proved by the same trigonometric identity for the tangent of the difference between two angles.

Finally, the middle portion of the front face of the prism that is covered by the spiral prism, shown in Figure 4(a), has area

$$\begin{aligned}
 P_c &= \int_0^H \sqrt{x^2(h) + y^2(h)} dh \\
 &= \int_0^1 \frac{1+t^2}{1+t} \frac{4H}{\pi} \frac{dt}{1+t^2} \\
 &= \frac{4H}{\pi} \int_0^1 \frac{dt}{1+t} = H 4 \ln 2 / \pi.
 \end{aligned} \tag{5}$$

From (3), (4), and (5), we see that $P_x + P_c + P_y = 2H$, the total area of the front face of the prism. This verification provides a check on our calculations.

Next, let us still look at the union of the prism and the spiral prism but focus on any one of the four lateral surfaces of the spiral prism bounded by the locus of two adjacent corners of the base square under the spiral transformation. See Figure 4(b). Here, the dotted lines trace the outline of two neighboring faces of the prism, and the solid lines show one lateral face of the spiral prism. The left part of this lateral surface is visible; the middle part is covered by the prism; and the right part is visible. The areas of these three parts are, respectively

$$\begin{aligned} S_y &= \int_0^H y(h) dh = H (1 - 2 \ln 2/\pi), \\ S_c &= \int_0^H \sqrt{x^2(h) + y^2(h)} dh = H 4 \ln 2/\pi, \\ S_x &= \int_0^H x(h) dh = H (1 - 2 \ln 2/\pi), \end{aligned} \tag{6}$$

with the total area of each lateral face of the spiral prism being $S_y + S_c + S_x = 2H$.

From (3)-(6), note that $S_y = S_x = P_x = P_y < S_c = P_c$, and $S_y + S_x = P_x + P_y > S_c = P_c$. Thus, each visible part has a smaller area than the covered part, but the two visible parts together have a larger area than the covered part. The proportion of each face of the prism (spiral prism) covered by the spiral prism (prism) is $P_c/(2H) = S_c/(2H) = 2 \ln 2/\pi = 0.4413$.

The total surface area of the union of the right prism and its spiral cousin is

$$4(P_x + S_x + S_y + P_y) = 16H (1 - 2 \ln 2/\pi),$$

which is a fraction $1 - 2 \ln 2/\pi = .5587$ of $16H$, and the total surface area of the intersection is

$$4(S_c + P_c) = 32H \ln 2/\pi,$$

which is a fraction $2 \ln 2/\pi = .4413$ of $16H$, the sum of the surface areas of the union and the intersection (or of the surface areas of the prism and the spiral prism).

3 Volumes of Various Parts

We already mentioned that the volume of the prism (and the spiral prism) is $4H$ unit³. The portion of the spiral prism outside the prism consists of four equivalent solids (two

of which are visible in Figure 2), each having volume

$$\begin{aligned}
 V_{S \setminus P} &= \int_0^H \frac{1}{2} x(h) y(h) dh \\
 &= \int_0^1 \frac{t}{1+t} (1-t) \frac{4H}{\pi} \frac{dt}{1+t^2} \\
 &= \frac{4H}{\pi} \int_0^1 \left(\frac{1}{1+t^2} - \frac{1}{1+t} \right) dt \\
 &= \frac{4H}{\pi} \left(\frac{\pi}{4} - \ln 2 \right) = H (1 - 4 \ln 2 / \pi).
 \end{aligned} \tag{7}$$

Likewise, the portion of the prism outside the spiral prism consists of four equivalent solids, each having volume

$$V_{P \setminus S} = \int_0^H \frac{1}{2} y(h) x(h) dh = H (1 - 4 \ln 2 / \pi). \tag{8}$$

It is a pleasant surprise that (7) and (8) indicate that the volumes of the parts of the prism (spiral prism) external to the spiral prism (prism) are the same. Consequently, the volume of the union is

$$4H + 4 \cdot H(1 - 4 \ln 2 / \pi) = 8H (1 - 2 \ln 2 / \pi),$$

which is a fraction $1 - 2 \ln 2 / \pi = .5587$ of $8H$, and the volume of the intersection is

$$4H - 4 \cdot H(1 - 4 \ln 2 / \pi) = 16H \ln 2 / \pi,$$

which is a fraction $2 \ln 2 / \pi = .4413$ of $8H$, the sum of the volumes of the union and the intersection (or of the sum of the volumes of the prism and the spiral prism).

4 Curve Lengths

We evaluated the lengths of three types of curves.

4.1 Lengths of the Separator Curves

When the prism and the spiral prism are superimposed, by symmetry and 180° rotation symmetry, each of the eight boundary curves on their lateral faces is equally long. What is that length?

We know the functional form of each separator curve (either $x(h)$ or $y(h)$). These curves are shown in Figure 4(a). Each curve has length $l_x = \int_0^H \sqrt{1 + (x'(h))^2} dh$ or $l_y = \int_0^H \sqrt{1 + (y'(h))^2} dh$. See [4], page 585, for the arc-length formula.

To evaluate l_y , note that $y(h) = 1 - \tan((h/H)\pi/4)$ implies that $y'(h) = -\sec^2((h/H)\pi/4) \pi/(4H) = -(1+t^2) \pi/(4H)$. Hence,

$$\begin{aligned} l_y &= \int_0^H \sqrt{1 + (y'(h))^2} dh \\ &= \int_0^1 \sqrt{1 + (1+t^2)^2 \left(\frac{\pi}{4H}\right)^2} \frac{4H}{\pi} \frac{dt}{1+t^2} \\ &= \int_0^1 \sqrt{\left(\frac{4H}{\pi(1+t^2)}\right)^2 + 1} dt. \end{aligned} \quad (9)$$

Similarly, we have

$$l_x = \int_0^H \sqrt{1 + (x'(h))^2} dh = \int_0^1 \sqrt{\left(\frac{4H}{\pi(1+t^2)}\right)^2 + \frac{4}{(1+t)^4}} dt. \quad (10)$$

As mentioned earlier, the two curves $x(h)$ and $y(h)$ exhibit a 180° rotation symmetry. Therefore, even though (9) and (10) have slightly different expressions, in fact, $l_x = l_y$. However, neither (9) nor (10) has a closed-form expression. The numerical values (computed with WolframAlpha, see [5]) for some choices of H are shown in Table 1. For large H , $x(h)$ and $y(h)$ being almost linear, their length is approximately $\sqrt{1 + H^2}$.

Table 1: The length of the curve between the prism and the spiral prism

H	1	2	3	4
$l_x = l_y$	1.4219	2.2442	3.1688	4.1284
$\sqrt{1 + H^2}$	1.4142	2.2361	3.1623	4.1231

4.2 Lengths of the Loci of Boundary Points

What is the length of $\mathcal{L}_A$, the locus traced by the corner A (or the length $\mathcal{L}_X$, the locus traced by any other point X) of the square base as it rotates and moves up?

Note that the point X is always at a constant (orthogonal) distance $g(X)$ from the axis of the spiral prism as it rotates and moves up. Here, $g(X)$ is the distance between X and the center O of the square base. Specifically, $g(X) = \sec \angle XOM$, where M is the

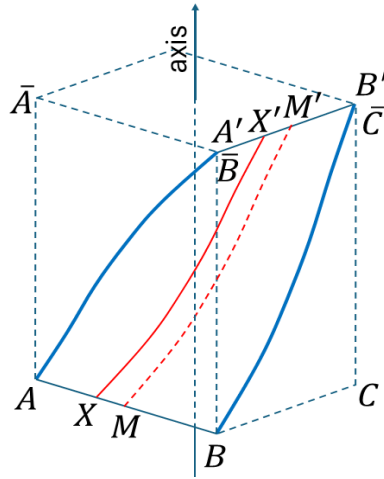


Figure 6: Locus of any point under uniform rotation and vertical translation

midpoint of the side of the square base on which X lives. Thus, the locus $\mathcal{L}_X$ is a spiral in 3D about the axis. As such, it lives on the lateral surface of a hypothetical quarter cylinder of radius $g(X)$ and height H . This lateral surface can be planarized into a rectangle of height H and width $g(X)\pi/2$, and the locus $\mathcal{L}_X$ can be identified with a diagonal of the rectangle. Hence, according to the Pythagorean theorem, its length is $\sqrt{H^2 + g^2(X)\pi^2/4}$.

Specifically, the length of the locus $\mathcal{L}_M$ of M is $l(M) = \sqrt{H^2 + \pi^2/4}$, and that of $\mathcal{L}_A$ is $l(A) = \sqrt{H^2 + \pi^2/2}$. If X is chosen uniformly over the side of the square base, then the expected length of the locus of X is

$$\begin{aligned} E[l(X)] &= \int_0^1 \sqrt{H^2 + (1+x^2)\pi^2/4} dx \\ &= \frac{1}{4\pi} \left[\pi\sqrt{4H^2 + 2\pi^2} - (4H^2 + \pi^2) \ln \left(\frac{\sqrt{4H^2 + 2\pi^2} - \pi}{\sqrt{4H^2 + \pi^2}} \right) \right] \end{aligned} \quad (11)$$

The closed-form expression in (11) comes from WolframAlpha. Also, $E[l^2(X)] = H^2 + \pi^2/3$, from which the standard deviation of the length of the locus of X equals $\sigma(l(X)) = \sqrt{E[l^2(X)] - (E[l(X)])^2}$. Some numerical values are given in Table 2.

4.3 Lengths of the Vertical Curves

When traveling vertically on the surface of the right prism, from any point on the bottom square to its vertical projection on the top square, each curve is a line segment H units tall. When traveling vertically on the surface of the spiral prism, what is the length

Table 2: The lengths of the loci of $\mathcal{L}_M$ and $\mathcal{L}_A$, and the expectation $E[l(X)]$ and the standard deviation $\sigma(X)$ of $l(X)$, when X is selected uniformly on a side of the base square

H	1	2	3	4
$l(M)$	1.8621	2.5431	3.3863	4.2974
$l(A)$	2.4361	2.9891	3.7329	4.5755
$E[l(X)]$	2.0639	2.6966	3.5041	4.3910
$\sigma(l(X))$	0.1741	0.1344	0.1040	0.0833

of the curve from any point on the bottom square to its vertical projection on the top square?

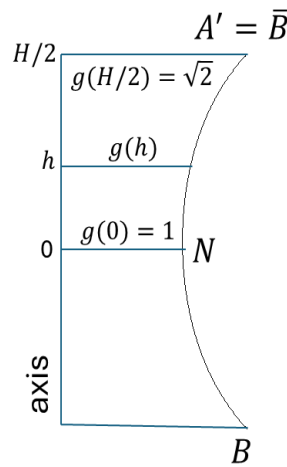


Figure 7: The curve joining a corner of the base to its vertical projection

Figure 7 shows the vertical curve $\mathcal{C}_{BA'}$ that joins B to $\bar{B} = A'$ (the end point of the locus $\mathcal{L}_A$, under uniform spiraling) when traveling on the surface of the spiral prism. Note that $B, A' = \bar{B}$ and the axis of the spiral prism are in a plane. Due to the vertical symmetry of $\mathcal{C}_{BA'}$ above and below its midpoint N at the mid-height of the spiral prism, it is enough to study the upper half of $\mathcal{C}_{BA'}$. So, cut the spiral prism in half by a horizontal plane at its mid-height.

Caution: Since we are now working with the upper half of the spiral prism, we redefine the parameters h, θ and t .

Consider the point Z on $\mathcal{C}_{BA'}$ at height $0 < h < H/2$. Let $g(h)$ denote its (orthogonal) distance from the axis. Clearly, $g(0) = 1$ and $g(H/2) = \sqrt{2}$. To find $g(h)$ for an arbitrary $h \in [0, H/2]$, note that the square cross-section at height h is rotated by an angle $\theta = \frac{h}{H/2} \frac{\pi}{4} = (h/H) \pi/2$ relative to the base of the upper half of the spiral prism.

Therefore, $g(h) = \sec(\theta) = \sec((h/H) \pi/2)$. Hence, writing $t = \tan \theta = \tan((h/H) \pi/2)$, we have

$$g'(h) = \sec\left(\frac{h\pi}{2H}\right) \tan\left(\frac{h\pi}{2H}\right) \frac{\pi}{2H} = t\sqrt{1+t^2} \frac{\pi}{2H},$$

and

$$dt = \sec^2\left(\frac{h\pi}{2H}\right) \frac{\pi}{2H} dh = (1+t^2) \frac{\pi}{2H} dh.$$

Hence, the vertical curve from B to $A' = \bar{B}$ has total length

$$\begin{aligned} l_{\text{vert}} &= 2 \int_0^{H/2} \sqrt{1 + (g'(h))^2} dh \\ &= \frac{4H}{\pi} \int_0^1 \frac{\sqrt{1 + \left(\frac{\pi}{2H}\right)^2 t^2(1+t^2)}}{1+t^2} dt \\ &= \int_0^1 \sqrt{\left(\frac{4H}{\pi(1+t^2)}\right)^2 + \frac{4t^2}{1+t^2}} dt. \end{aligned} \tag{12}$$

Like (9) and (10), (12) does not have a closed-form expression. The numerical values (computed with WolframAlpha) for some choices of H are shown in Table 3. Of course, they are greater than $2N\bar{B}$, the Euclidean distance (in 3D) between the midpoint N of $\mathcal{C}_{BA'}$ and A' .

Table 3: The length of the vertical curve $\mathcal{C}_{BA'}$ on the surface of the spiral prism

H	1	2	3	4
l_v	1.3757	2.2314	3.1640	4.1262
$2 * N\bar{B}$	1.0824	2.0424	3.0285	4.0214

Let P be any point (not necessarily a corner) in segment AB of the bottom square base. What is the length of the vertical curve $\mathcal{C}_{P\bar{P}}$ joining P to its projection $\bar{P}$ on the top square that always travels on the surface of the spiral prism? The surprising answer is given below.

Lemma 1: For any P , the length of the vertical curve $\mathcal{C}_{P\bar{P}}$ is exactly l_{vert} .

Proof: Let Z be the intersection point between $\mathcal{C}_{P\bar{P}}$ and $\mathcal{L}_A$. Horizontally slice the spiral prism through Z , and stack the bottom part on top (matching the base square to the top square and P to $\bar{P}$). Then the curve $\mathcal{C}_{P\bar{P}}$ becomes the curve $\mathcal{C}_{Z\bar{Z}}$, which, being identical to $\mathcal{C}_{BA'}$, has length l_{vert} . $\square$

We conclude the paper with a more astounding result stated in Lemma 2. Figure 7 shows that the axis of the spiral prism and the curve $\mathcal{C}_{B,\bar{B}}$ are on a plane, with orthogonal distance between them given by $g(h)$, $h \in [-H/2, H/2]$.

Lemma 2: If you revolve the curve $\mathcal{C}_{B,\bar{B}}$ about the axis, the volume of the solid of revolution is $4H$, the same as the volume of the spiral prism (and the prism)!

Proof: Transform the spiral prism as follows: (1) Cut the spiral prism into wedges with vertical plane cuts through the axis in radial increments of $\Delta\theta$. (2) Cut each such wedge horizontally through the point farthest from the axis (at a distance $\sqrt{2}$ but at varying heights). (3) For each wedge, stack the lower part atop the upper part and bottom-align the two stacked parts. None of these transformations changes the volume of the solid.

Taking the limit as $\Delta\theta \rightarrow 0$, you have converted the spiral prism into the solid of revolution, with volume

$$\begin{aligned} V_{\text{rev}} &= 2\pi \int_0^{H/2} g^2(h) dh = 2\pi \int_0^{H/2} \frac{dh}{\cos^2(\frac{h\pi}{2H})} \\ &= 2\pi \int_0^1 (1+t^2) \frac{2H}{\pi} \frac{dt}{1+t^2} = 4H, \end{aligned}$$

completing the proof. □

The same is true if you revolve the curve $\mathcal{C}_{P,\bar{P}}$ about the axis, for any point P on the base square. The proof is similar to the proofs of Lemma 1 and Lemma 2.

5 Summary and an Open Problem

We measured surface areas, solid volumes, and curve lengths exhibited by a prism, a spiral prism, their union, and intersection. Whereas we worked with a spiral prism on a square base, the methods extend to a spiral prism on a regular n -gonal base. College Mathematics teachers may assign such tasks to their bright students.

Given any two points on the lateral surfaces of a right prism (not a spiral prism), with the base square and the top square missing, the geodesic (or the shortest path on the lateral surface) between them is found as follows: planarize the lateral faces into a rectangle (duplicating the face containing one point); join the duplicated points to the other given point with lines; and choose the shorter of these lines. Its length is found by using the Pythagorean theorem.

However, the lateral faces of the spiral prism cannot be planarized. We leave open the problem of finding the geodesic (or the shortest path on the lateral surface) between any two points on the spiral prism, with the base square and the top square missing, and measuring its length.

Acknowledgment

Thanks to my wife for choosing a strange vase for her creeper. A very special thanks to Ms. Monica Thompson-Deal for making the physical model, shown in Figure 2(b), which I can exhibit during a talk. Thanks to my colleague Dr. Bogdan Nica for some discussions on the problem.

Bibliography

- [1] Wikipedia, the Free Encyclopedia, Twisted Prism,
[https://en.wikipedia.org/wiki/Prism_\(geometry\)#Twisted_prism](https://en.wikipedia.org/wiki/Prism_(geometry)#Twisted_prism)
- [2] Wikipedia, the Free Encyclopedia, Cavalieri's Principle,
https://en.wikipedia.org/wiki/Cavalieri's_principle
- [3] Jyotirmoy Sarkar. Watch pi appear and disappear magically. *International Journal of Mathematics, Statistics and Operations Research*, 3(2), 301-326.
- [4] James Stewart, *Calculus*, 8 ed., 2016, Boston, MA: Cengage Learning.
- [5] WolframAlpha: From the Makers of Wolfram Language and Mathematica.
<https://www.wolframalpha.com/>



Jyotirmoy Sarkar's research areas include enumeration, probability, statistics, and reliability theory. He enjoys reading, 'riting, 'rithmetic and R-coding.

9 Derksen's Proof of the Fundamental Theorem of Algebra¹

J. K. Verma

Department of Mathematics, IIT Gandhinagar

Email: jugal.verma@iitgn.ac.in

1 Introduction

The fundamental theorem of algebra asserts that any non constant polynomial with complex coefficients has a complex root. The first rigorous proof was given in Gauss' 1799 thesis. Gauss published three more proofs. Since then many proofs have been given. Most of the known proofs are included in the book [3]. For a brief history of the theorem, we refer the reader to an excellent article by R. Remmert in the book [1].

2 Harm Derksen's proof of the fundamental theorem of algebra

We shall present a simplified version of Harm Derksen's proof published in 2003 in the American Mathematical Monthly [2]. This proof uses eigenvectors of commuting linear operators on real and complex vector spaces. The basic idea behind the proof is that the roots of a polynomial can be realised as eigenvalues of the companion matrix of the polynomial. Indeed, consider a polynomial with complex coefficients:

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.$$

¹These are notes of a lecture delivered in the *Training Programme in Mathematics* during 3-4 June, 2018 at NISER, Bhubaneswar.

The quotient space $V = \mathbb{C}[x]/(f(x))$ is a finite dimensional complex vector space. Let $[g]$ denote the residue class in V of $g \in \mathbb{C}[x]$. Then the set $\mathcal{B} = \{[1], [x], \dots, [x^{n-1}]\}$ is a basis of V .

Proposition 1. *The polynomial $f(x)$ is the characteristic polynomial of the linear operator*

$$m_x : V \longrightarrow V \text{ given by } m_x([g]) = [xg].$$

Proof. The matrix of m_x with respect to $\mathcal{B}$, called the **companion matrix of $f(x)$** is

$$C(f(x)) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

□

Lemma 2.1. *The roots of $f(x)$ are eigenvalues of m_x and the characteristic polynomial of $C(f(x))$ is $f(x)$.*

Proof. Let r be a root of $f(x)$. Then $f(x) = (x - r)g(x)$ for some $g(x) \in \mathbb{C}[x]$. Then $m_x([g(x)]) = [xg(x)] = r[g(x)]$. Hence r is an eigenvalue of m_x . Conversely, if r is an eigenvalue of m_x then there is a nonzero $[g(x)]$ such that $m_x([g(x)]) = r[g(x)]$. Hence $xg(x) - rg(x) = h(x)f(x)$ for some $h(x) \in \mathbb{C}[x]$. It is clear that $x - r \nmid h(x)$. Hence $f(r) = 0$. □

Using this observation, we see that the FTA follows once it is proved that any operator $T : V \longrightarrow V$ acting on a finite dimensional complex vector space V has an eigenvector and an eigenvalue. The surprising idea of Derksen is to show that commuting operators have a common eigenvector and hence an eigenvalue. To show this, we first show that commuting operators acting on real vector spaces of odd dimension have a common eigenvector.

Proposition 2. *Let V be a real vector space of odd dimension and $A, B : V \longrightarrow V$ be commuting linear operators. Then they have a common eigenvector.*

Proof. Apply induction on $n = \dim V$. Let $n = 1$. Choose any nonzero $v \in V$. Then $V = \{\alpha v \mid \alpha \in \mathbb{R}\}$. Hence all nonzero vectors in V are eigenvectors of A and B . Suppose

now the theorem is true for all odd dimensional vector spaces of dimension at most $n-2$. The characteristic polynomial $\chi_A(x)$ of A has real coefficients and it has odd degree. By the intermediate value theorem, we have a real root λ of $\chi_A(x)$. Consider the following real vector subspaces of V .

$$K = \text{Ker}(A - \lambda I) \text{ and } R = \text{Im}(A - \lambda I).$$

Since A and B commute, the subspaces K and R are invariant under B . Indeed, let $v \in R$. Then $v = (A - \lambda I)(u)$ for some $u \in V$. Therefore

$$Bv = B(Au - \lambda u) = A(Bu) - \lambda Bu = (A - \lambda I)Bu \in R.$$

If $v \in K$ then

$$(A - \lambda I)Bv = B(A - \lambda I)v = 0.$$

Since $\dim K \geq 1$, either $K = V$ or K is a proper subspace of V . If $K = V$ then B has an eigenvector in K . If K is proper, then by the rank-nullity theorem $\dim K + \dim R = n$. Therefore either K or R is odd dimensional. By the induction, hypothesis, A and B will have a common eigenvector either in K or in R . $\square$

Definition 2.2. Let $P(K, d, r)$ denote the assertion that any r commuting linear operators on a K -vector space V of dimension n where $d \nmid n$ have a common eigenvector.

The proof of the above result can easily be modified to prove the following

Proposition 3. *If $P(K, d, 1)$ is true then $P(K, d, 2)$ is true.*

Proposition 4. *$P(\mathbb{C}, 2, 2)$ is true.*

Proof. It is enough to prove that $P(\mathbb{C}, 2, 1)$ is true. Let $\dim V = n$ where n is odd and $T : V \rightarrow V$ be a linear operator. Let A be the matrix of T with respect to any basis of V . Consider the real vector space

$$\mathcal{H} = \{B \in \mathbb{C}^{n \times n} \mid B \text{ is Hermitian}\}.$$

Then $\dim \mathcal{H} = n^2$ is odd. Consider the following operators defined on $\mathcal{H}$.

$$\phi(B) = \frac{AB + BA^*}{2} \text{ and } \psi(B) = \frac{AB - BA^*}{2i}.$$

The maps ϕ and ψ commute. Therefore they have a common eigenvector, say B . Let $\phi(B) = \lambda B$ and $\psi(B) = \mu B$. Then the equation

$$AB = \phi(B) + i\psi(B) = (\lambda + i\mu)B$$

shows that all nonzero column vectors of B are eigenvectors of A . $\square$

Theorem 1. $P(\mathbb{C}, 2^k, 2)$ is true.

Proof. Apply induction on k . We have already proved the $k = 1$ case. Suppose that $P(\mathbb{C}, 2^r, 2)$ is true for $r = 1, 2, \dots, k - 1$. Let $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a linear operator where $2^{k-1} \mid n$ but $2^k \nmid n$. Consider the complex vector space

$$S_n = \{B \in \mathbb{C}^{n \times n} \mid B \text{ is skew-symmetric}\}.$$

Then $\dim S_n = \binom{n}{2}$. Hence $2^{k-1} \nmid \dim S_n$. Define the linear maps $L_1, L_2 : S_n \rightarrow S_n$ by

$$L_1(B) = AB - BA^t \text{ and } L_2(B) = ABA^t.$$

Then L_1 and L_2 commute. Hence there is a common eigenvector, say $B \in S_n$. Then

$$L_1(B) = AB - BA^t = \lambda B \text{ and } L_2(B) = ABA^t = \mu B.$$

$$\text{Therefore } L_2(B) = \mu B = ABA^t = A(AB - \lambda B) = A^2B - \lambda AB.$$

Hence $(A^2 - \lambda A - \mu I)B = 0$. Let $\alpha, \beta \in \mathbb{C}$ be the roots of $X^2 - \lambda X - \mu = 0$. Then

$$(A - \alpha I)(A - \beta I)B = 0.$$

If $(A - \beta I)B = 0$ then any nonzero column vector of B is an eigenvector of A . If $(A - \beta I)B \neq 0$, then any nonzero column vector of $(A - \beta I)B$ is an eigenvector of A . $\square$

Theorem 2. If $S, T : V \rightarrow V$ are commuting linear operators on a complex vector space of dimension n then they have a common eigenvector.

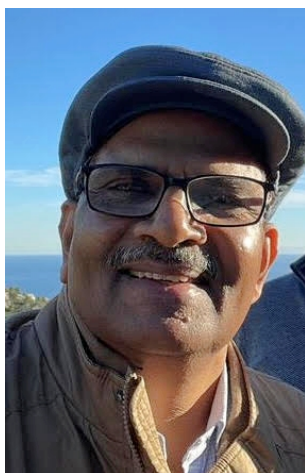
Proof. Let $n = \dim V$. Then there exists a positive integer k such that 2^k does not divide n . Since $P(\mathbb{C}, 2^k, 2)$ is true, the theorem follows. $\square$

Acknowledgment. I thank Sudeshna Roy for reading an earlier version very carefully.

Bibliography

- [1] H.-D. Ebbinghaus et al. *Numbers*, Graduate Texts in Mathematics 123, Springer, 1991.

- [2] Harm Derksen. The fundamental theorem of algebra and linear algebra. *Amer. Math. Monthly*, 110, (2003), no. 7, 620-623.
- [3] Benjamin Fine and Gerhard Rosenberger, *The Fundamental Theorem of Algebra*, Springer, 1997.



Jugal Verma is a visiting professor at IIT Gandhinagar since 2024. Prior to this he served in the department of Mathematics at IIT Bombay during 1990-2024. He is a fellow of the Indian National Science Academy (New Delhi) and the National Science Academy (Allahabad).

His research interests are mainly in commutative algebra. He likes to popularise mathematics by various means such as organising workshops for teachers and researchers, writing expository articles for college and university students and delivering lectures in various outreach programs.

10 To Identify, or not to Identify, that is the Question

S. Kesavan

Chennai Email: kesh@imsc.res.in

Abstract

Banach spaces and their duals are fundamental objects of study in Functional Analysis. A celebrated theorem of F. Riesz implies that a Hilbert space can be identified with its dual. However, when dealing with more than one space at a time, especially with continuous inclusion of one in another, one cannot simultaneously identify each space with its dual. This is explained below. In a lighter vein, we use an analogy from *Vedanta*.

Introduction

In a branch of Indian philosophy, called *Vedanta*, there are three important schools of thought. All of them deal with the inter-relationship of the soul, called *atman*, or *jivatma*, and the divine, called *brahman* or *paramatma*.

The *Advaita* philosophy, propounded by *Adi Sankaracharya*, says that these two are one and the same, while the *Dvaita* philosophy of *Madhvacharya* says that they are different entities. In the English language, the former is called *monism*, while the latter is called *duality*. The third school of thought, spearheaded by *Ramanujacharya*, is called *Visishtadvaita*, and is also known as *qualified monism* or *monism with distinction*. Roughly speaking, it says that while it is possible to identify *atman* and *brahman*, terms and conditions apply.

Thyagaraja, the prolific nineteenth century composer in Carnatic music, seeks the guidance of Vishnu to understand the nuances of these theories in his composition *Dvaitamu sukhama? Advaitamu sukhama?* (in the raga *Reetigowla*).

A similar situation occurs in mathematics too, especially in Functional Analysis. Given a normed linear space V (the base field being $\mathbb{R}$ or $\mathbb{C}$), we have the associated dual space V^* , of all continuous linear functionals on V . The study of either gives a lot of information on the other. Since the dual space is always complete, let us also assume that V is complete, i.e., V is a Banach space, so that both spaces are on the same footing for comparison.

Dvaita

In most Banach spaces, though closely related, V and V^* are as different as chalk and cheese in appearance. For instance, the dual of the space $\mathcal{C}[0, 1]$ of continuous functions on the interval $[0, 1]$, is a space of measures. In some spaces, they are similar in structure, but still not isomorphic to each other. This occurs in the case of the Lebesgue spaces $L^p(\mu)$, where μ is a measure on a suitable measurable space. A particular case of these are the sequence spaces ℓ_p . In the case of the Lebesgue spaces, if $1 < p < \infty$, then the dual of $L^p(\mu)$ is the space $L^q(\mu)$, where q , the conjugate exponent, is defined by the relation

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The structures of these are very similar but the spaces are not isometrically isomorphic. There is one exception, namely the case $p = 2$, when they coincide.

Advaita

A Hilbert space is a Banach space whose norm comes from an inner-product. In the case of Hilbert spaces, we have an important result known as the *Riesz representation theorem*, which is stated as follows (cf. Kesavan [1]).

Theorem 1. *Let H be a Hilbert space. Then given any $\varphi \in H^*$, there exists a unique $x \in H$ such that, for all $y \in H$, we have*

$$\varphi(y) = (y, x),$$

where $(\cdot, \cdot)$ denotes the inner-product in H . Further, $\|\varphi\| = \|x\|$. ■

Let us assume that the base field is $\mathbb{R}$, so that the inner-product is symmetric. In this case, thanks to the Riesz representation theorem, the mapping $\varphi \mapsto x$ defines an isometric isomorphism from H^* onto H . (In the complex case, the map is conjugate

linear.) Thus we can identify H and H^* . This is what happens when $p = 2$ in the case of the Lebesgue spaces mentioned above.

Remark: The mathematician F. Riesz was a past master in the computation of duals of Banach spaces. Apart from the theorem stated above, all the results mentioned in the previous section (Dvaita) go under the name Riesz representation theorem!

Visishtadvaita

Thanks to the Riesz representation theorem stated above, when working with (real) Hilbert spaces, we have no difficulty in identifying the space with its own dual. However, can we do this indiscriminately in all situations? The answer is negative and it is a consequence of the following result.

Theorem 2. *Let V and H be two (real) Hilbert spaces whose norms and inner-products are respectively denoted by $\|\cdot\|_V, (\cdot, \cdot)_V$ and $\|\cdot\|_H, (\cdot, \cdot)_H$. Assume that V is a subspace of H and that the inclusion is continuous, i.e., there exists a constant $C > 0$ such that, for all $v \in V$, we have*

$$\|v\|_H \leq C\|v\|_V.$$

Assume, further, that as a subspace of H , the space V is dense in H . Then H^ is a subspace of V^* and the inclusion is continuous and dense.*

Proof: We can summarize the properties of V and H using the notation

$$V \xhookrightarrow{d} H.$$

Thus, we have to show that

$$H^* \xhookrightarrow{d} V^*.$$

Let $\varphi \in H^*$. This is a linear functional on V as well. Further,

$$|\varphi(v)| \leq \|\varphi\|_{H^*} \|v\|_H \leq C\|\varphi\|_{H^*} \|v\|_V.$$

This shows that φ is a continuous linear functional on V and that

$$\|\varphi\|_{V^*} \leq C\|\varphi\|_{H^*}.$$

If two elements of H^* agree on V , then, by the density of the inclusion of V in H , it follows that they agree on all of H , i.e., they are equal. This shows that H^* is indeed a subspace of V^* and the preceding inequality shows that this inclusion is continuous.

Since Hilbert spaces are reflexive, the only continuous linear functionals on V^* are of the form

$$J(v)(\varphi) = \varphi(v), \text{ for all } \varphi \in V^*,$$

where $v \in V$. Assume that there is an element of V^{**} which vanishes on H^* . Thus for some $v \in V$, we have that

$$\varphi(v) = 0, \text{ for all } \varphi \in H^*.$$

By the Riesz representation theorem, it follows that, for all $x \in H$, we have $(v, x)_H = 0$. Since V is contained in H , it follows that

$$\|v\|_H^2 = (v, v)_H = 0,$$

from which it follows that $v = 0$. Thus, if a continuous linear functional on V^* vanishes on H^* , that functional is zero. By the Hahn-Banach theorem, it follows that H^* is dense in V^* . This completes the proof. ■

Now, let us use the Riesz representation theorem and identify H with its dual, H^* . We then get

$$V \xrightarrow{d} H \cong H^* \xrightarrow{d} V^*.$$

Now it is obvious that if we simultaneously identify V with its dual V^* , we will get absurd results.

Thus, when dealing with two Hilbert spaces, with continuous and dense inclusion of one in another, we cannot invoke Riesz' theorem for *both* of them. We generally use the Riesz identification only for the larger space, which, in this situation, is called the *pivot space*.

The above situation occurs frequently in Functional Analysis. A notable example is that of the Sobolev spaces, which play a key role in the modern study of partial differential equations.

Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, for any positive integer m , we define the Sobolev space $H^m(\Omega)$ as the space of all functions in $L^2(\Omega)$ (i.e., the space $L^2(\mu)$, where μ is the N -dimensional Lebesgue measure on Ω), such that all its partial derivatives (in the sense of distributions) up to order m are also in $L^2(\Omega)$. This is a Hilbert space and contains, in particular, all C^∞ functions with compact support inside Ω , as a subspace. The closure of this subspace in $H^m(\Omega)$ is denoted $H_0^m(\Omega)$, and, being closed, is a Hilbert space as well. We have the inclusions

$$\dots \subset H^{m+1}(\Omega) \subset H^m(\Omega) \subset \dots \subset H^2(\Omega) \subset H^1(\Omega) \subset L^2(\Omega)$$

and

$$\dots \subset H_0^{m+1}(\Omega) \subset H_0^m(\Omega) \subset \dots \subset H_0^2(\Omega) \subset H_0^1(\Omega) \subset L^2(\Omega).$$

All the above inclusions are continuous and dense. We identify the (pivot) space $L^2(\Omega)$ with its dual. The duals of the Sobolev spaces are not identified with these spaces. In particular, the dual of $H_0^m(\Omega)$ is a space of distributions denoted $H^{-m}(\Omega)$. We thus have the continuous and dense inclusions

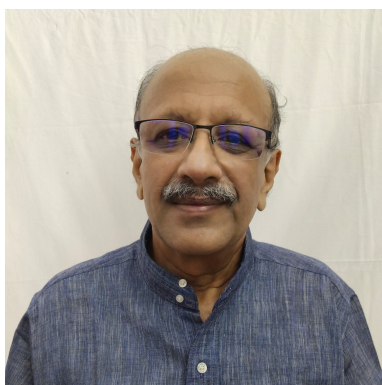
$$\cdots \subset H_0^2(\Omega) \subset H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) \subset H^{-2}(\Omega) \subset \cdots.$$

(Why $H^{-m}(\Omega)$ denotes the dual of $H_0^m(\Omega)$ and not the dual of $H^m(\Omega)$ is another story and is based on the theory of distributions.) Thus, though the Riesz representation theorem allows every (real) Hilbert space to be identified with its dual, when dealing with a tower of spaces with continuous and dense inclusions, we keep the spaces and their duals distinct, except for the pivot space.

For details on the theory of distributions, Sobolev spaces and their applications to partial differential equations, see, for instance, Kesavan [2].

Bibliography

- [1] Kesavan, S. *Functional Analysis*, Second Edition, TRIM 52, Hindustan Book Agency, 2022.
- [2] Kesavan, S. *Topics in Functional Analysis and Applications*, Third Edition, New Age International (P) Limited, Publishers, 2019.



Kesavan retired as Professor from the Institute of Mathematical Sciences, Chennai. His research interests include, Partial Differential Equations, Homogenization and Isoperimetric Inequalities. He is the author of five books. Texts of his popular articles can be found in his homepage: www.imsc.res.in/~kesh. His courses on Functional Analysis, Measure and Integration and Sobolev Spaces and PDE can be found on the NPTEL platform.