TECHNOLOGY IN MATHEMATICS EDUCATION

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GOALS OF MATHEMATICS EDUCATION

What are the goals of mathematics education?

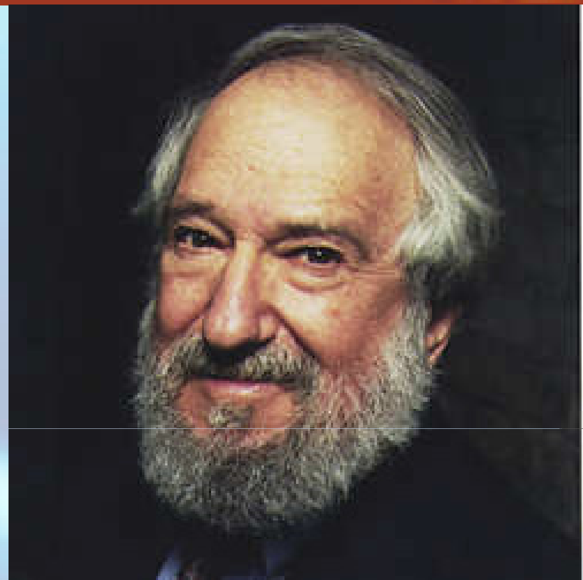
Simply stated, there is one main goal – Mathematisation of the child's thought processes – NCF 2005

Mathematics is, after all, a human activity……….
The pupil himself should reinvent mathematics. During this process, the learner is engaged in an activity where experience is described, organized and interpreted by mathematical means. This activity is mathematising.

Hans Freudenthal

SEYMOUR PAPERT

Seymour Papert vividly talked about children using computers as instruments for learning and for enhancing creativity, innovation, and "concretizing" computational thinking.



The essence of computational thinking is what we can do while interacting with computers, as extensions of our mind, to create and discover (Knuth, 1980).

From reports and curriculum documents

According to the ACM Pathways Report 2013

"By 2020, every one of two jobs in STEM fields will be in computing"

The NGSS (Next Generation Science Standards)

- Emphasizes the need to infuse computational thinking in high school math.
- Highlights the growing importance of computation and digital technologies across scientific disciplines.

The Common Core guidelines

"students should be able to use technological tools to explore and deepen their understanding of concepts"

The Pedagogical challenge

- How do we realise computational thinking in the mathematics classroom?

What is computational thinking…..

Computational thinking skills (Weintrop et.al, 2016)

- The ability to deal with open ended problems
- Persistence and confidence in dealing with complexity
- Representing ideas in computationally meaningful ways
- Breaking down problems into simpler ones
- Creating abstractions for aspects of the problem at hand
- Assessing strengths and weaknesses of a representation
- Generating algorithmic solutions

Motivation for CT in math classroom:

The reciprocal relationship – using computation to enrich mathematics learning through technology and using mathematical contexts to enrich CT.

RECOMMENDATIONS OF NCF 2005

School Mathematics

- Activity oriented
- Accessible to all and affordable by all
- Emphasis on relevance of mathematics

Shift from content to processes

- Formal problem solving, Estimation and approximation, Use of patterns
- Visualization, Reasoning and proof
- Making connections, Mathematical communication
- Explorations and investigatory problems involving CT have great relevance here.

Technology

- Taking over computations
- Reducing cognitive load
- Opportunities to explore concepts numerically, graphically, symbolically

TECHNOLOGY FOR MATHEMATICS INSTRUCTION

DYNAMIC GEOMETRY SOFTWARE

- GEOGEBRA
- CINDRELLA
- DrGeo
- GEOMETRIC EXPLORER
- GEOMETER'S SKETCHPAD
- CABRI/3D

HANDHELD TECHNOLOGY

- GRAPHIC CALCULATORS
  - FX CG50
- ALGEBRAIC CALCULATORS
  - CLASSPAD 300

COMPUTER ALGEBRA SYSTEMS (CAS)

- MATHEMATICA
- MAPLE
- MATLAB
- DERIVE
- MAXIMA

SPREADSHEETS

- MS EXCEL
- CALC
- LIBREOFFICE CALC

Learning theories support this view of learning

Vygotsky's Sociocultural theory

- learners should be provided with socially rich environments in which they can explore knowledge domains with their fellow students, teachers and outside experts

Piaget's theory of constructivism

- Learning is constructed, not transmitted, and that learners play an active role in the learning process. Technology can be a great enabler in this regard.

Cognitive aspects of technology

Technology helps to develop our thinking.

- Amplifier – restructures our thinking and increases the range of activities that we can perform in a limited time. Gives us access to higher level concepts.

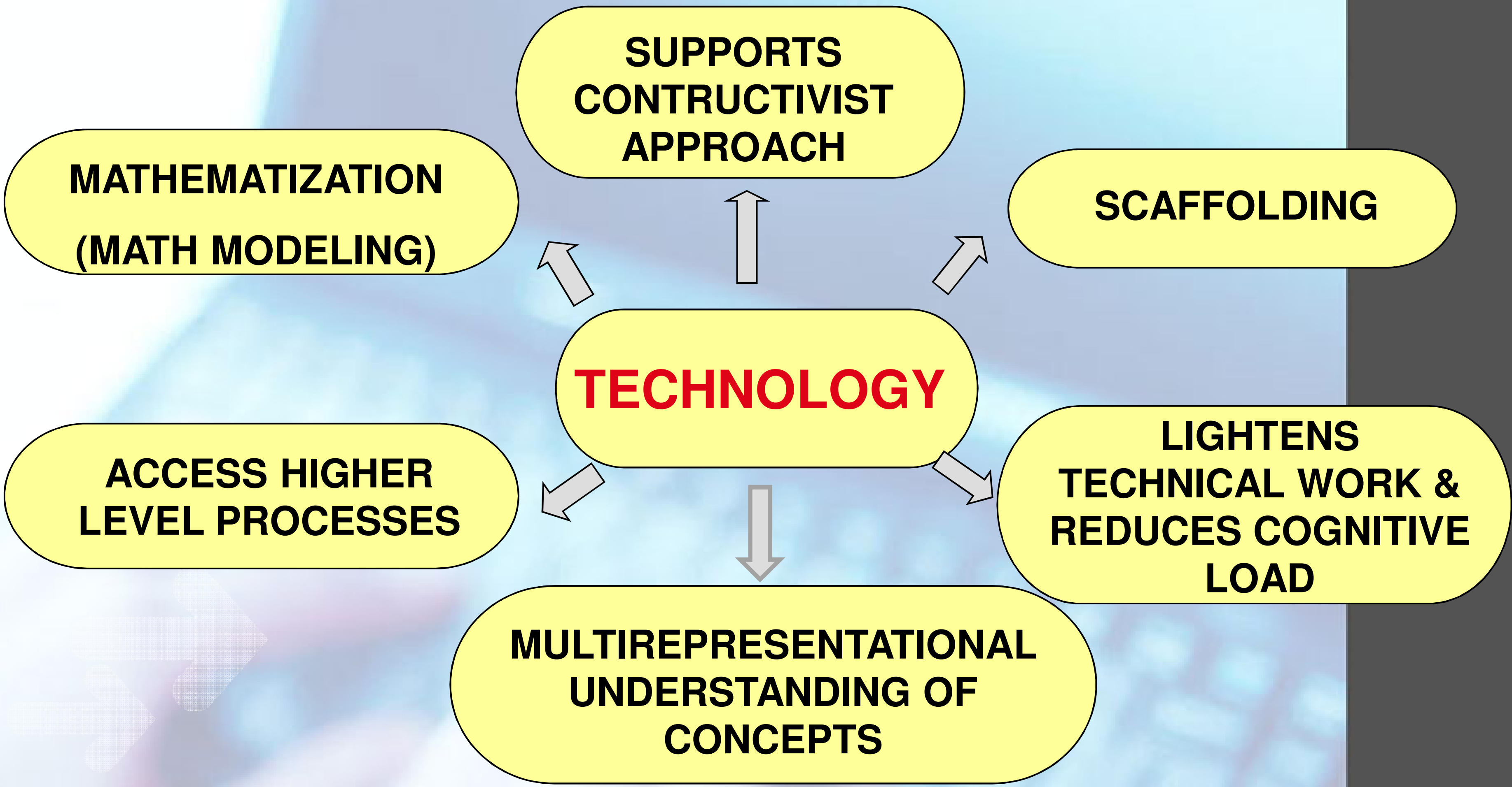
- Reorganiser – Emphasises processes such as search for patterns, problem solving etc and enables learning through exploration and discovery - a sequence different from the traditional ones.

Theory of Instrumental Genesis

Raberdel and Verillion

- Tool mediated mathematical thinking
- The process through which students (and teachers) develop their capacity to make use of a tool for mathematical purposes.
- Users develop mental schemes which transform the tool from being a material artifact to a functional instrument.
- Instrumentation: The user turns a tool into an instrument for performing a specific task by developing a scheme.
- Instrumentalization: Specific functionalities are attributed by the user to the tool (not necessarily intended by the designer of the tool)

TECHNOLOGY : A CATALYST FOR REALIZING THE GOALS OF MATHEMATICS EDUCATION



Projects – Examples from the math classroom

Project topic	Mathematical concepts	Investigations
Applications to Genetics	Matrices and probability	Predict the genotype distribution of a plant population after any number of generations.
Cryptography (RSA Algorithm, Hill Cipher)	Number theory, Matrices	Modifying the method so that cracking the algorithm becomes more difficult, wrote programs in C++
Fractals	Geometric Sequences, Self-similarity, fractional dimension, Iteration and recursion	Creating new fractal curves and patterns and analysing them using GS and spreadsheets.
Newton's method and fractal patterns	Newton's method (elementary calculus), convergence of sequences	Exploring the nth roots of unity, plotting the Newton's basins of attraction and observing fractal patterns
Queuing models	Differential equations, birth-death processes	Analysing queuing models at a petrol pump and fast food counter and suggesting methods to optimise profits a nd reduce waiting time.
Fourier series and applicatio ns	Understanding Gibbs phenomenon	To reduce the impact of Gibbs phenomenon
Cellular Automata	Combinatorics, algorithms	Classifying the 256 ECA, encryption, creating new CA

APPLICATION OF MATRICES TO GENETICS

Autosomal inheritance

- the inherited trait under consideration (say petal color in a certain plant) is assumed to be governed by a set of two genes, denoted by **A** (red color) and **a** (white color)

- Three genotypes: AA (red), Aa (pink), aa (white)

- Students were asked to list all possible parent pairings along with the probabilities of the resulting offspring combinations.

		Parent pairings					
		AA - AA	**AA - Aa**	**AA - aa**	**Aa - Aa**	**Aa - aa**	**aa-aa**
Offspring outcomes	**AA**	**1**	**1/2**	**0**	**1/4**	**0**	**0**
	Aa	**0**	**1/2**	**1**	**1/2**	**1/2**	**0**
	aa	**0**	**0**	**0**	**1/4**	**1/2**	**1**

APPLICATION OF MATRICES TO GENETICS

Investigations

Explore the example of autosomal inheritance to *create a model which could predict the genotype distribution of a plant population after any number of generations* under specific breeding programs.

- For example, what happens when all plants are fertilised with plants of genotype AA (red flowers)

- with plants of genotype Aa (pink flowers)

APPLICATION OF MATRICES TO GENETICS

The modeling process

a_n, b_n and c_n: fraction of plants of genotypes **AA**, **Aa** and **aa** in the n th generation,

$$a_n + b_n + c_n = 1 \ \forall n$$

If all plants are fertilised with AA

$$a_n = a_{n-1} + \frac{1}{2} b_{n-1} \qquad b_n = \frac{1}{2} b_{n-1} + c_{n-1} \qquad c_n = 0$$

$$x^{(n)} = Mx^{(n-1)} \qquad x^{(n)} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} \qquad x^{(n-1)} = \begin{bmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{bmatrix} \qquad M = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

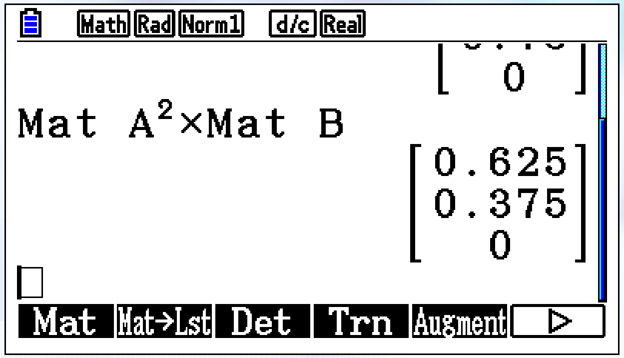
$$x^{(n)} = M^n x^{(0)}$$

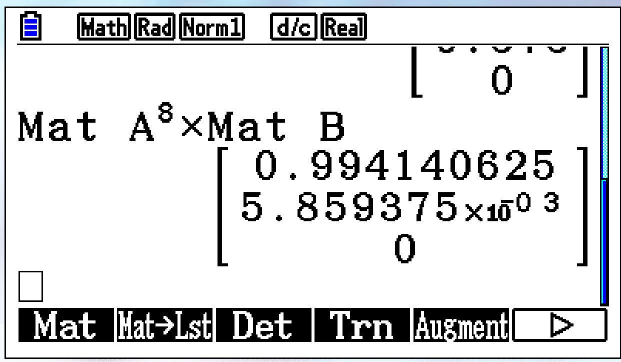
the genotype distribution of a plant population after any number of generations, given the initial distribution.

Computations using the CASIO cg 50 GDC or Excel

Students calculated the *steady state distributions of the three genotypes* under specific breeding situations such as

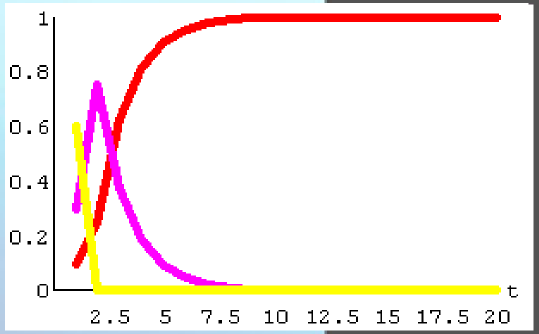
- If each plant was fertilised with type Aa.
- If each plant was fertilised with a plant of its own genotype.
- If alternate generations of plants were fertilised with genotypes AA and Aa respectively.





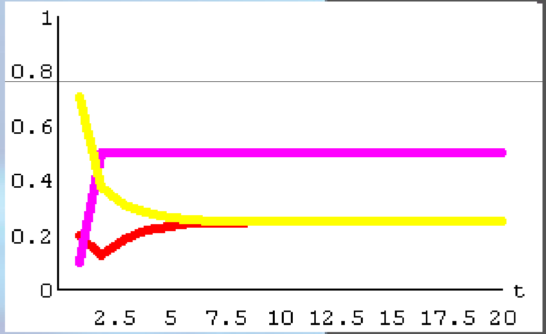
EXPLORATIONS WITH MATHEMATICA

Case 1: All plants fertilised with AA. In steady state all plants will have red flowers.



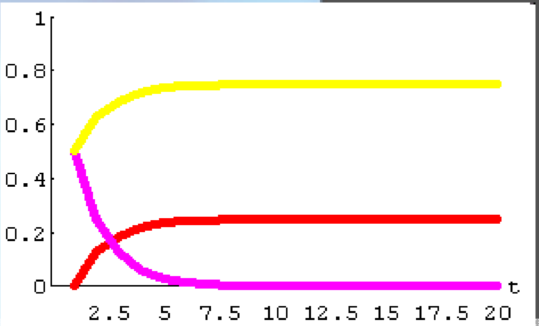
Case 2: All plants fertilized with Aa,

25% - AA, 50% - Aa, 25% - aa



Case 3: All plants fertilized with aa

50% - aa, 50% - AA



STUDENTS' EXPLORATIONS WITH MATHEMATICA

All Plants Fertilized With	Initial Distribution	Steady State		
		AA	Aa	aa
AA	[0 .5 .5]	100	0	0
	[.3 .3 .4]	100	0	0
Aa	[0 .5 .5]	25	50	25
	[.3 .3 .4]	25	50	25
aa	[0 .5 .5]	0	0	100
	[.3 .3 .4]	0	0	100
Own Genotype	[.5 0 .5]	50	0	50
	[.5 .5 0]	75	0	25
	[0 .5 .5]	25	0	75

Newton's method for approximating the zeros of any polynomial is

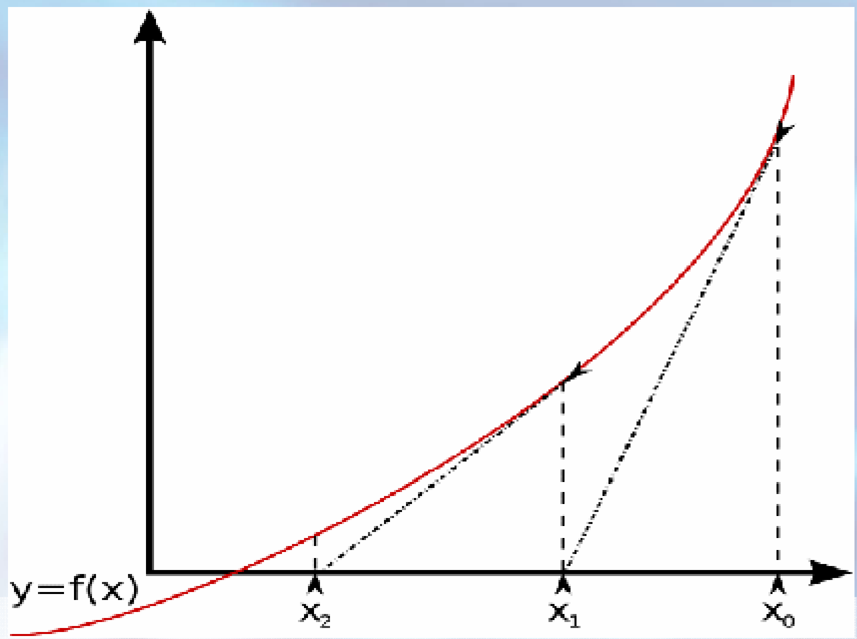
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Define Newton's function

$$N(z) = z - \frac{f(z)}{f'(z)}$$

The iterates $N^n(z_0)$

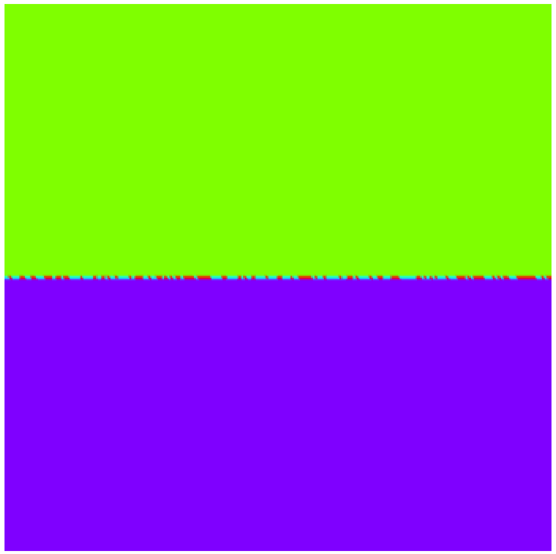
in general converge to a zero of f(z).



NEWTON'S METHOD AND FRACTAL PATTERNS

Newton's iterates of $f(z) = z^2 + 1$ converge to the roots i and –i. Mathematica was used to plot the basins of attraction B(i) and B(-i). These are the half planes created by the line L which is the perpendicular bisector of the line segment joining the zeros i and –i in the complex plane.

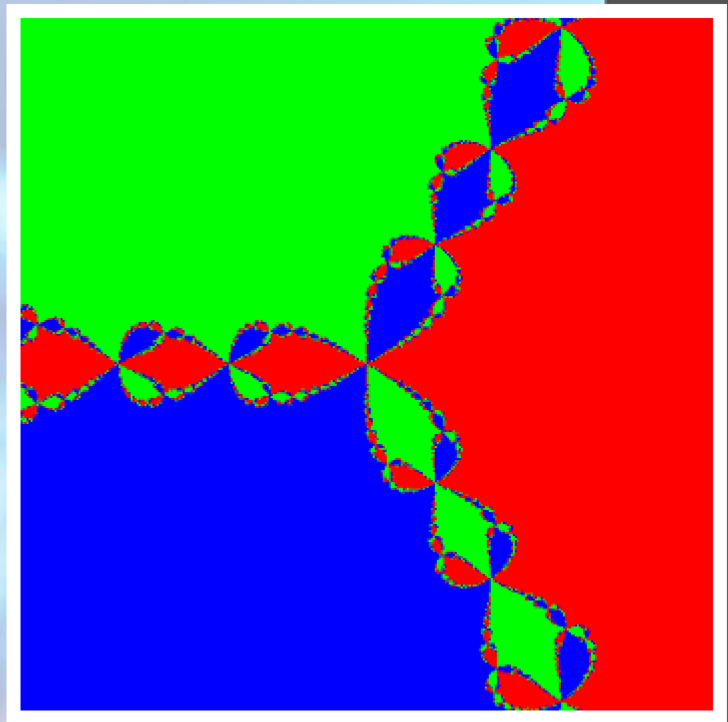
```
In[4]:= newton[z_] := z - f[z] / f'[z]
        plot[r_] := ListDensityPlot[Arg@FixedPoint[newton, #, 50] & /@
            Table[i + j I, {j, -r, r, 2 r / 365.}, {i, -r, r, 2 r / 365.}],
          ColorFunction → Hue,
          DataRange → {{-r, r}, {-r, r}}]
        f[z_] := z^4 - 1; plot[2.0];
```



NEWTON'S METHOD AND FRACTAL PATTERNS

For $f(z) = z^3 - 1$, its newton's iterates converge to the cube roots of unity.

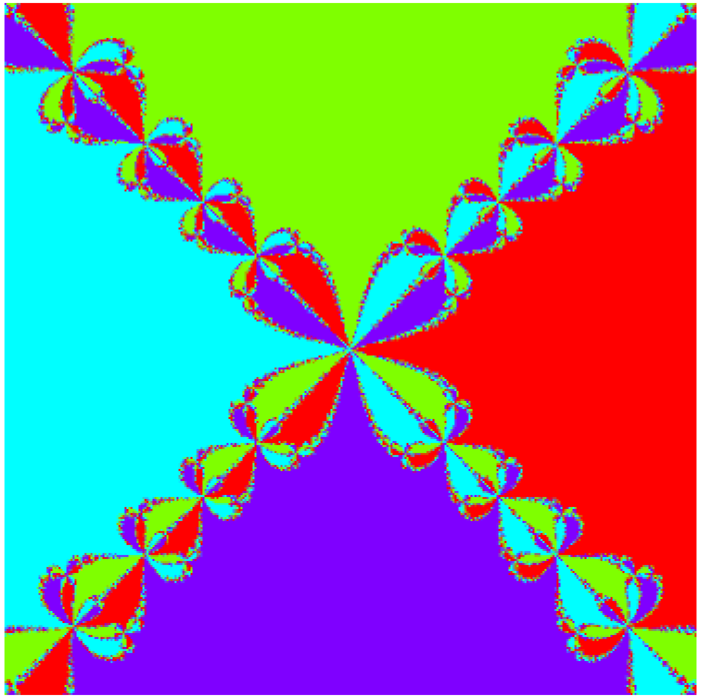
Enlarging the bulb to the left of the origin we get to see a fractal structure.

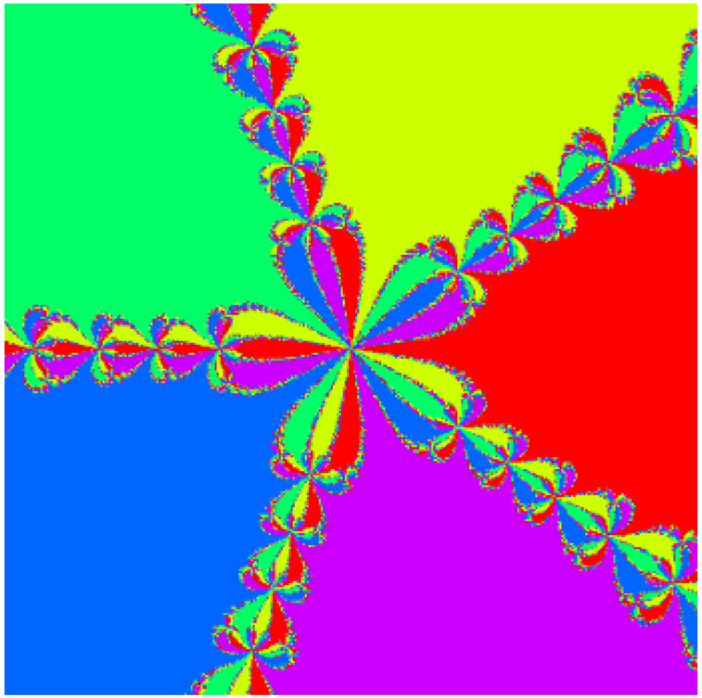


NEWTON'S METHOD AND FRACTAL PATTERNS

For $f(z) = z^4 - 1$, its newton's iterates converge to the fourth roots of unity, 1, -1, i, -i

For $f(z) = z^5 - 1$, its we see the rich structure of the Newton's basins of attraction





TEACHING OF GEOMETRY

The teaching of geometry is based on the use two registers

- register of language: language is a means of describing geometrical objects and relationships between and within them.

- register of diagrams: diagrams are two - dimensional representations of theoretical (ideal) objects which highlight graphical – spatial properties of objects. It is possible to read the properties of the theoretical object simply by looking at the diagram.

AFFORDANCES OF DGS

In a Dynamic geometry software (DGS)

- geometrical figures are dynamic and can be manipulated rather than static pictures on paper.
- parts of a figure can be dragged in the geometry window and its measurements will change dynamically in the algebra window.
- we can observe properties which remain invariant and those which vary. This helps us to verify properties and make conjectures.
- Researchers describe a DGS as a 'microworld' which provides the user rich opportunities to make and test conjectures.

Theory of Variation

.....when engaging in mathematical activities or reasoning, one often tries to comprehend abstract concepts by some kind of "mental animation", i.e. mentally visualizing variations of conceptual objects in hope of "seeing" patterns of variation or invariant properties.

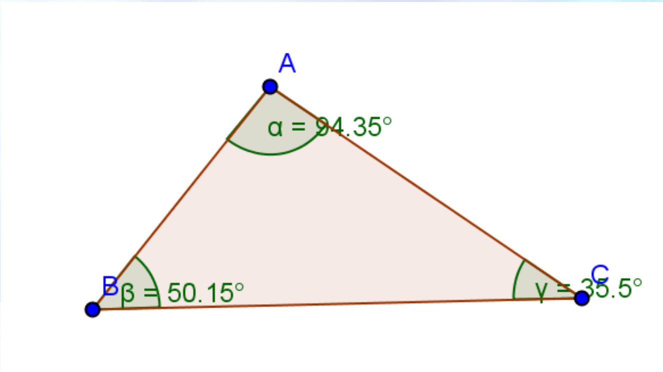
..... one of DGE's power is to equip us with the ability to retain (keep fixed) a background geometrical configuration while we can selectively bring to the fore (via dragging) those parts of the whole configuration that interested us in a mathematical thinking episode.

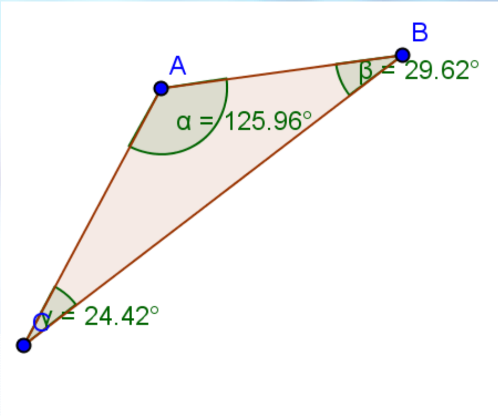
Discernment in DGS

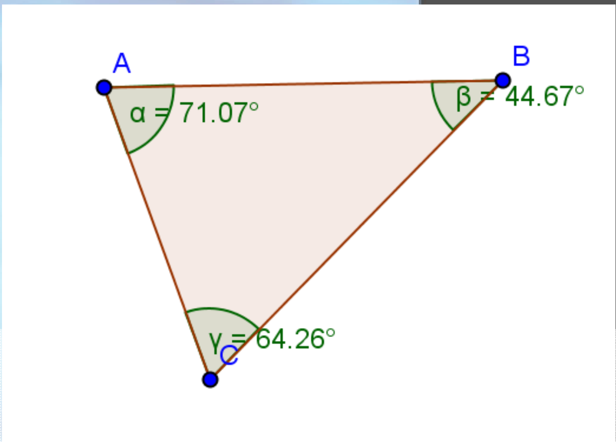
In DGE it is possible to discern critical invariant properties of geometrical objects under a continuous variation of certain components of the object.

For example, we may construct a Δ ABC, measure the interior angles and also the sum of all three angles.

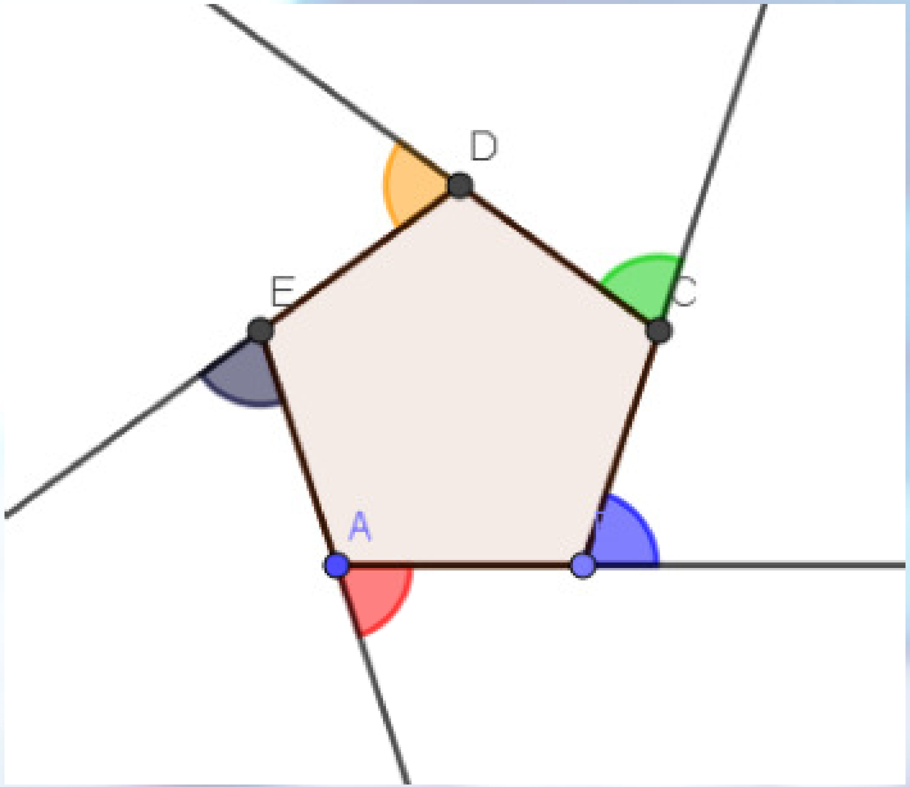
Dragging the vertex C will allow us to experience variation. Twofold interdependence – visual consistency of $\angle$ ABC + $\angle$ ACB + $\angle$ BAC = 180° will come to the fore in the midst of variation.

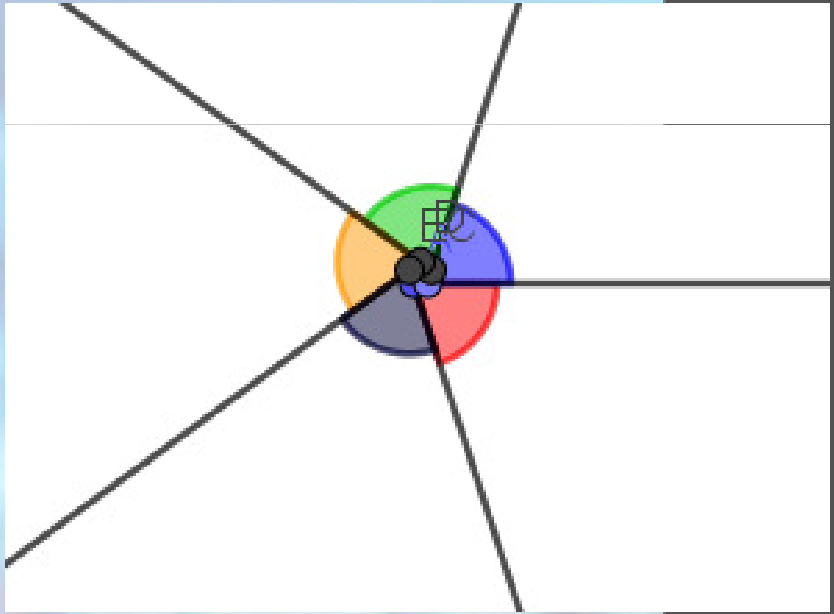






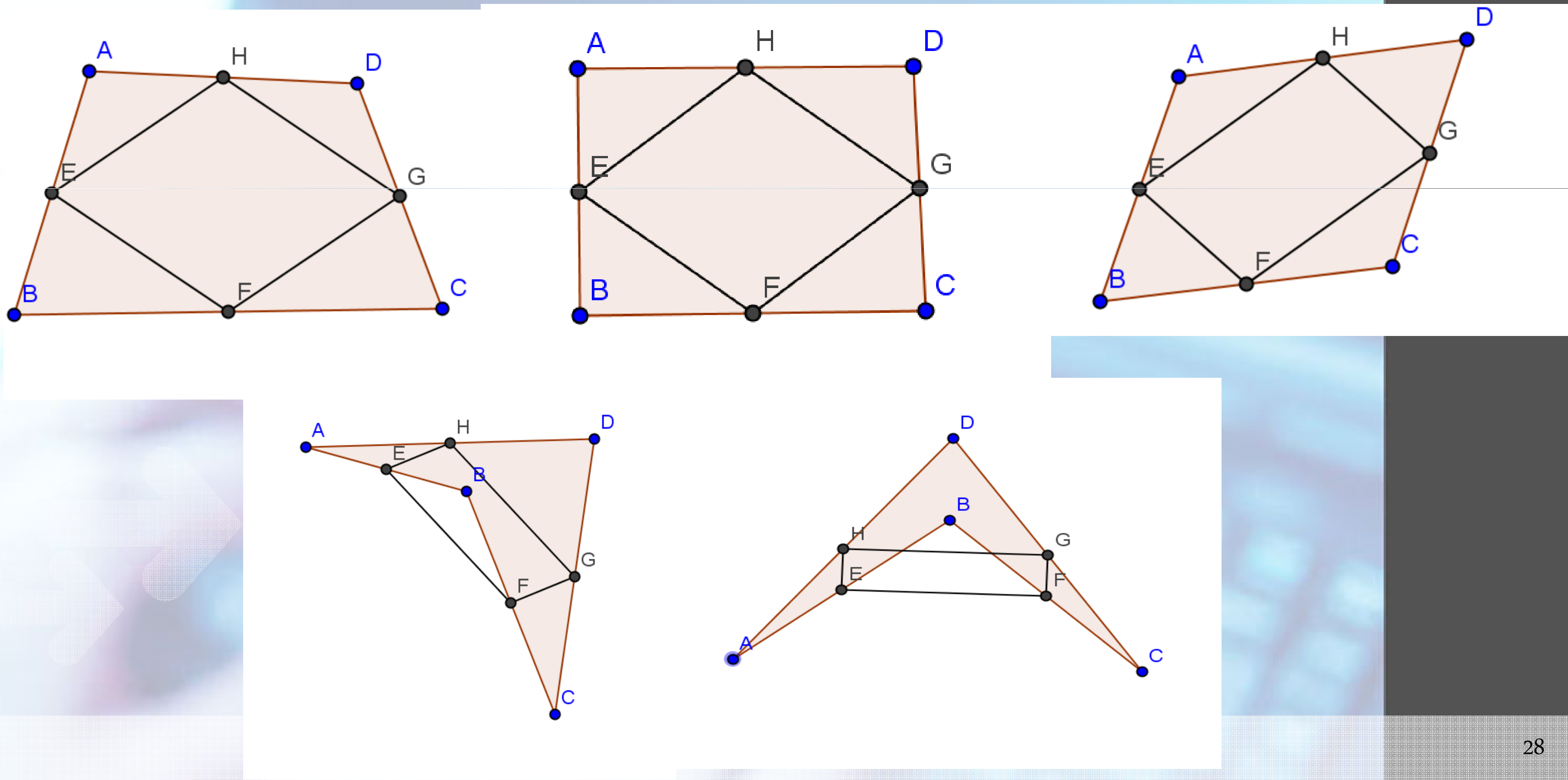
Exterior Angle Property of a Polygon



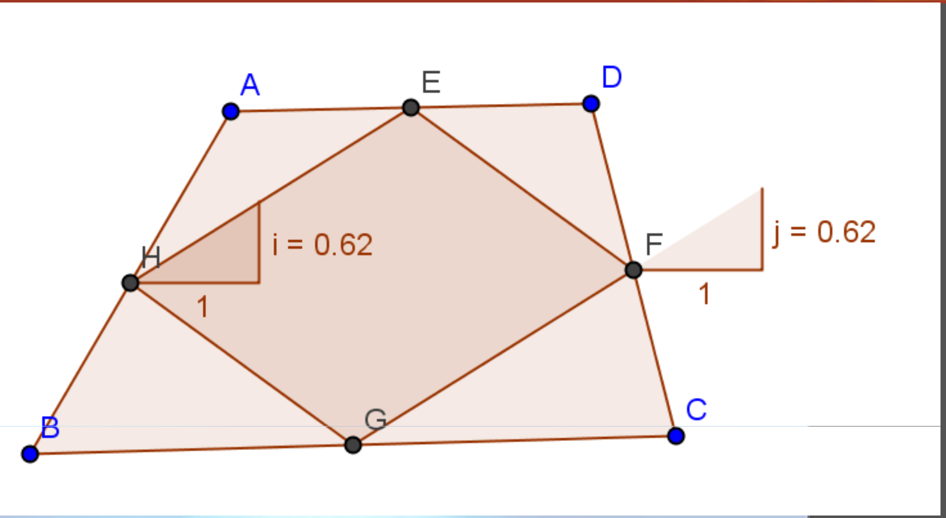


Dragging enables conjecture making:
A Research Study

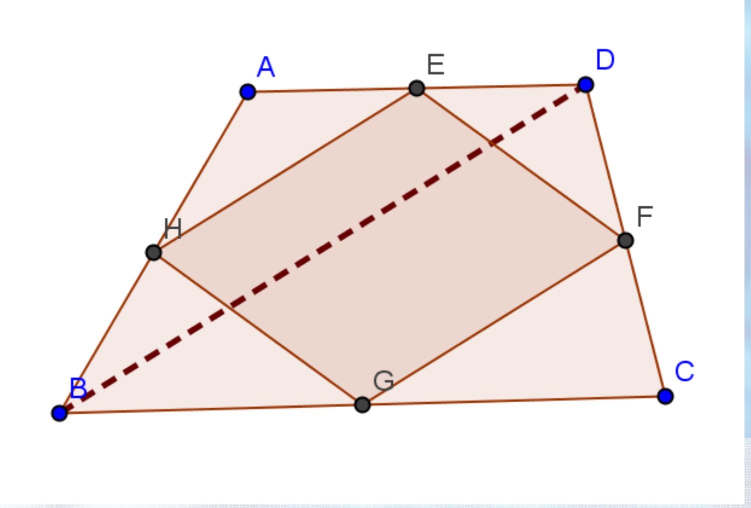
7th grade students of India and Sweden used GeoGebra to make conjectures about midpoint quadrilaterals



Concept of slope of a line was introduced and it was used to verify that opposite sides of EFGH are parallel.



The midpoint theorem was introduced and students applied it on ABCD to prove that EFGH is a parallelogram

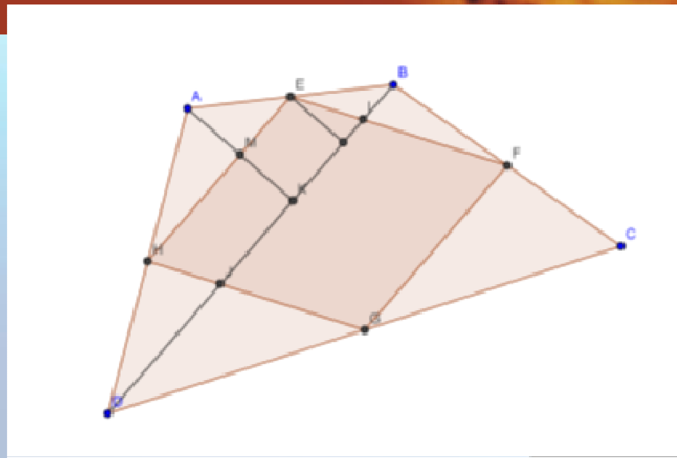


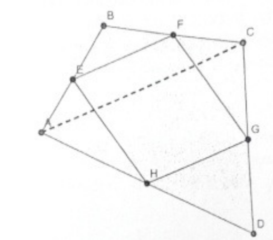
DGS: CONJECTURE TO PROOF

Students were asked to make a conjecture regarding the area of parallelogram EFGH in relation to quadrilateral ABCD

Used Algebra view to conclude that the area of EFGH is half that of the original quadrilateral.

Researchers: What does diagonal BD of quadrilateral ABCD do to the parallelogram EFGH?





The diagonal AC divides the quadrilateral ABCD into two triangles. In each of these triangles one of the sides of the midpoint quadrilateral is a mid segment. Use this information to validate the conjecture made by you in 9.

Note: $EF = \frac{1}{2} AC$, $HG = \frac{1}{2} AC$ and parallel to AC

It is parallel to AC

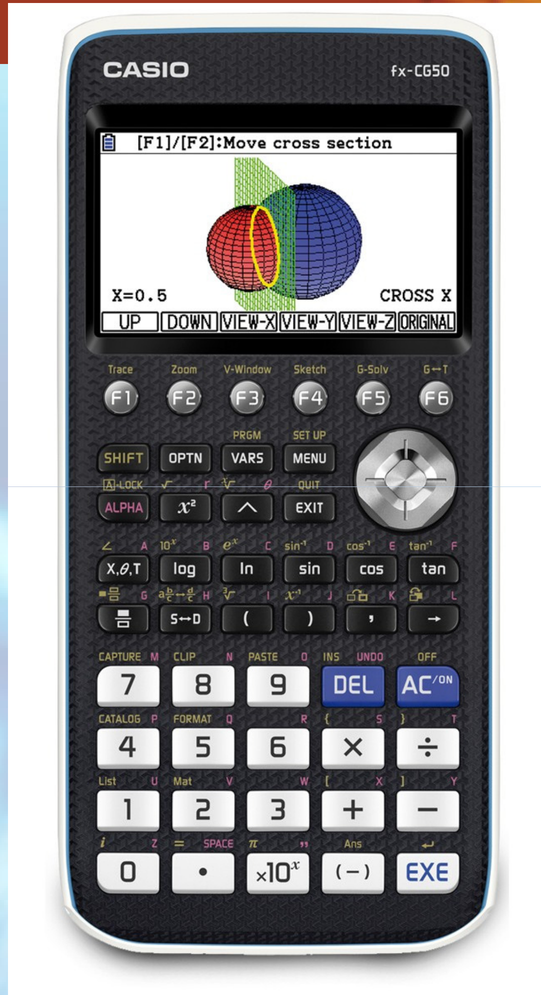
so EF || HG and equal lengths

So EF || FG

so EHGF is a parralogram

ROLE OF HAND HELD TECHNOLOGY (GRAPHIC CALCULATORS)

- Handheld portable devices which provide dynamic interactive working environments.
- Can be used in the classroom using the manager plus software. No separate lab is required.
- Have significant graphic, symbolic and numeric capabilities. (Matrices, complex numbers, symbolic differentiation, data analysis and programming).
- Cheap and affordable as compared to computer software.



Casio fx – cg 50 colour graphics calculator

Approach problems numerically

Exactness is overemphasized in school mathematics.

'Solve' should mean 'determine the solution with a prescribed degree of accuracy.'

1. When will your money triple in value at 6% interest compounded annually?

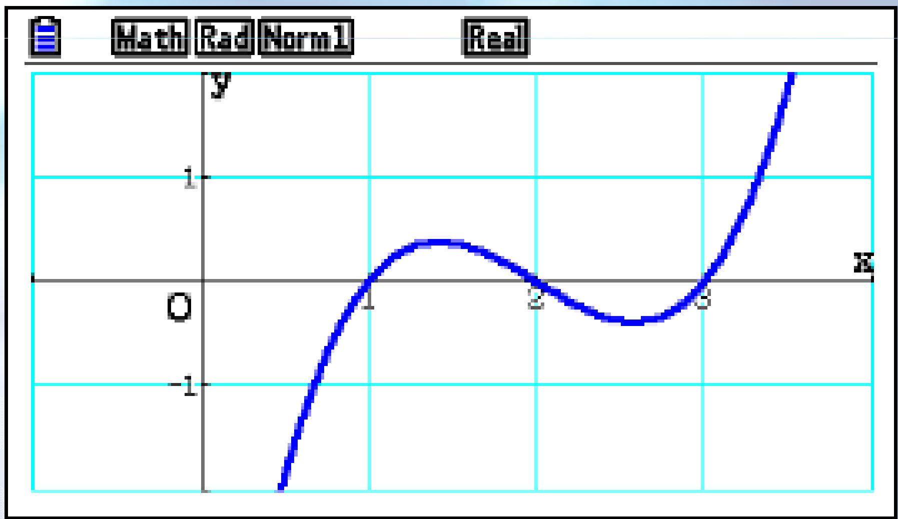
2. Solve the equation $x + 2^x = 7$

The calculator allows us to obtain the answer using guess, check and improve strategy.

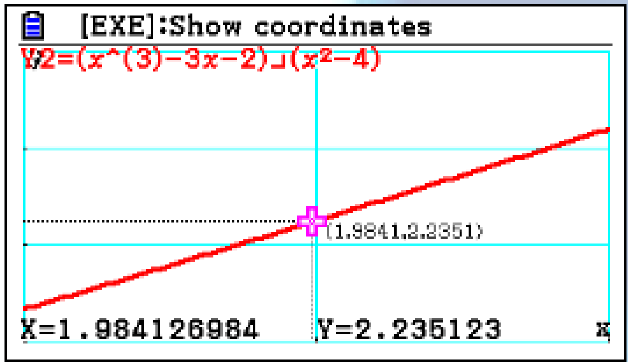
Visually support paper and pencil algebra

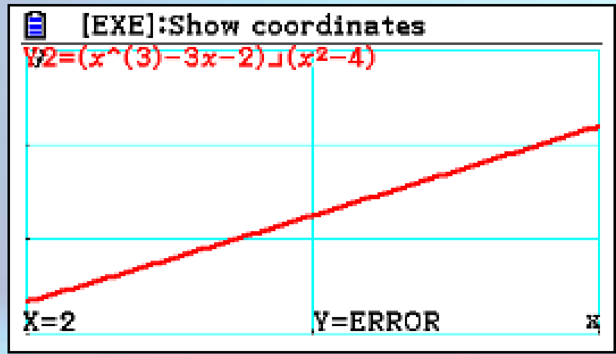
(Do algebraically and support visually)

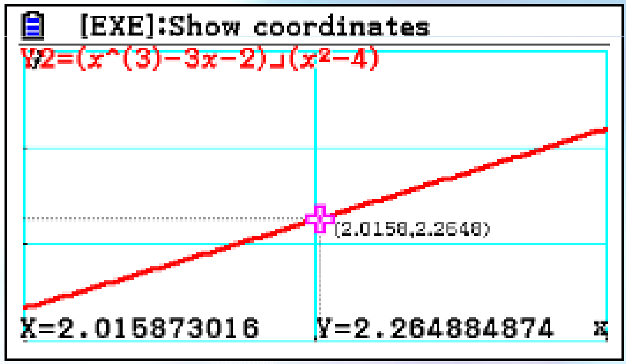
Solve the inequality $x^3 - 6x^2 + 11x - 6 < 0$

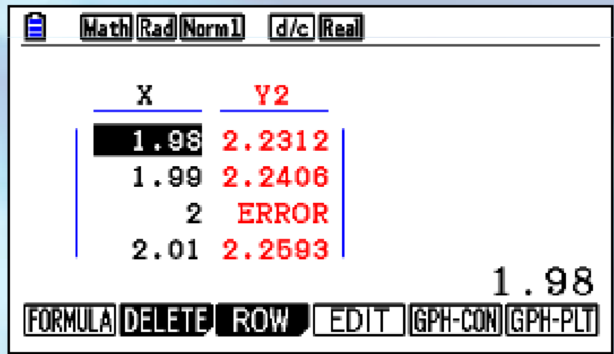


Multiple Representation of concepts





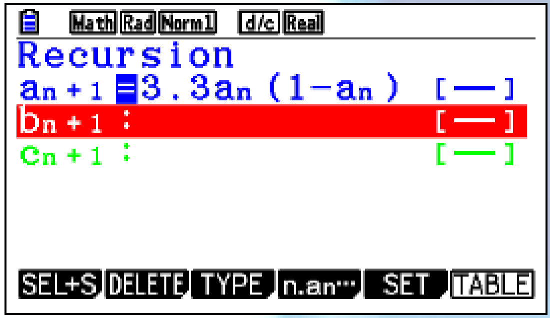


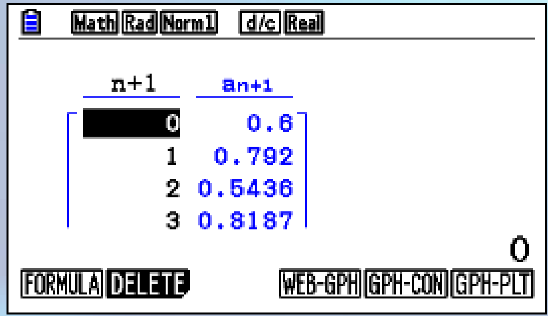


Screen shots of the graphics calculator displaying the limit graphically, and numerically.

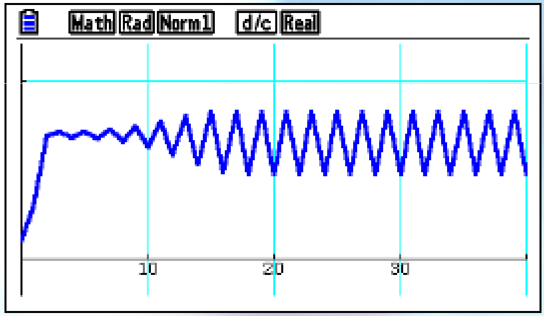
$$\lim_{x \to 2} \frac{x^3 - 3x - 2}{x^2 - 4} = \frac{9}{4}$$

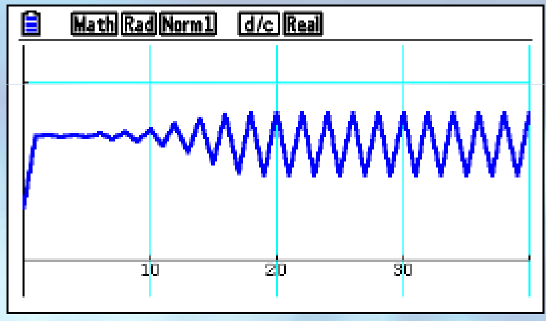
Modelling and Applications

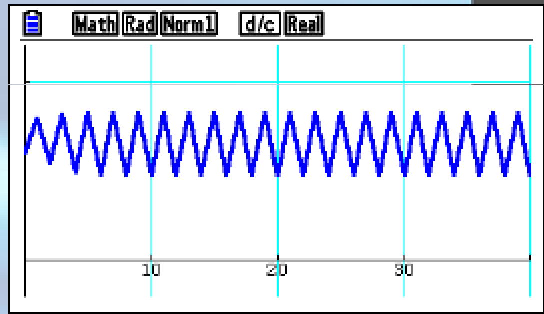


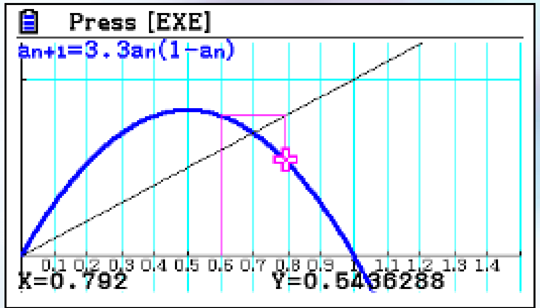


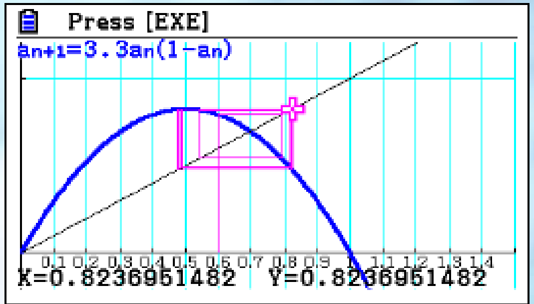
$f(x) = ax(1 - x)$





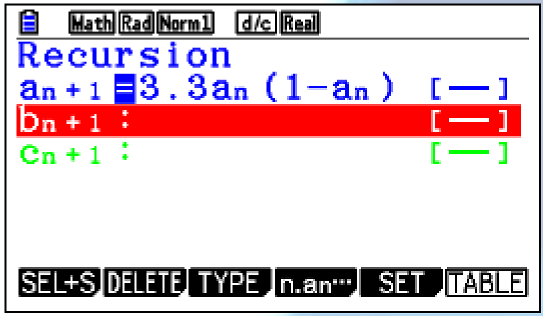




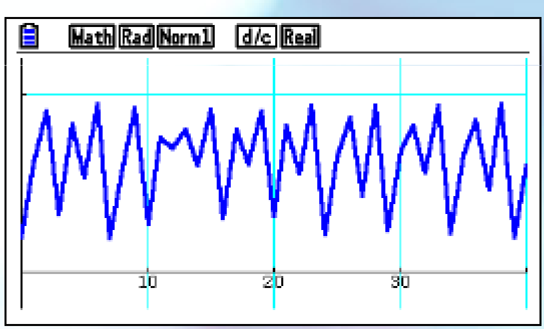


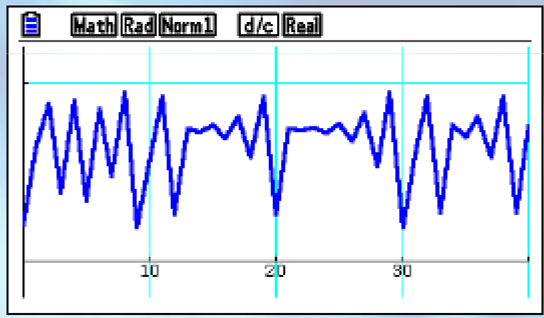
Understanding recursion numerically and graphically through dot and cobweb plots

Modelling and Applications

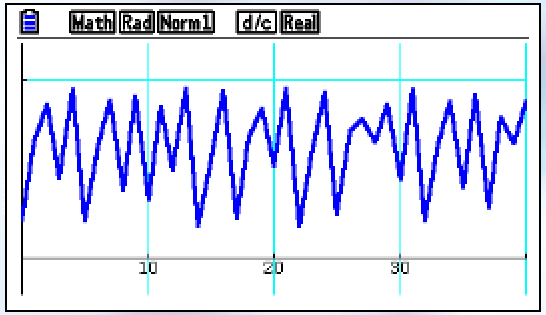


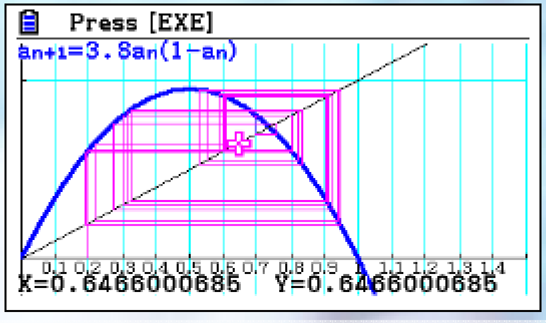
Visualising the onset of chaos and sensitivity to initial conditions





A slight change in the initial condition produces very different behaviour





Math Projects – a rich environment for CT

- Excellent student feedback – they owned their projects and were responsible for their learning, it encouraged group work.
- Technology was an ampifier – gave them access to higher level concepts, increased the scope of activities.
- Technology as reorganiser – the same sequence did not work for all projects.
- The outcomes were not predefined, there was a sense of 'suspense' and many 'aha' moments.Students asked many 'what if' questions.
- Presented projects in Fairs and Competitions.
- Engaged in processes which elicit CT
  - selecting between representations and creating new ones
  - simplifying or generalising problems
  - making conjectures
  - generating new questions for exploration
  - developed their own programs and used the outputs for further exploration

CONCLUSION

Implications for the curriculum

The inclusion of technology enabled CT explorations will imply

- Inclusion of mathematical modeling and applications, computer programming and discrete math in school curriculum
- Redesigning of the curriculum and pedagogy to bring about shift from content to processes.
- Considered use of appropriate technology to help restore balance between the need for computational skills and the need for experiencing processes such as exploration and conjecture.
- Large scale orientation of teachers to a pedagogy that inculcates computational thinking.

ROLE OF TECHNOLOGY

- Exploratory tool: served the purpose of a 'mathematical investigation assistant' giving students control over what they were learning.

- Computational tool: facilitated the computational process by quickly generating graphs and table of values. Helped to lighten the technical work so that students could focus on making observations and developing insight.

- An 'amplifier' : giving students access to a higher level concepts e.g In the Hill Cipher method.

- Supporting Paper and pencil methods: students did computations manually and verified using the tools. Technology gave meaning to their computations.

A Quote

Before computers there were very few good points of contact between what is most fundamental and engaging in mathematics and anything firmly planted in everyday life. But the computer — a “mathematics - speaking being” in the midst of the everyday life of the home, school, and workplace—is able to provide such links. The challenge to education is to find ways to exploit them.

Seymour Papert - Mindstorms