
Blackboard
Issue 1
MTA (I)
Here is the Ramanujan-Hardy formula for the calculation of the number of partitions:
1/ 2+1/ 3+1/7 +1/ 43+1/ 1807+ ... = 1

Contents

1 Editorial

This is the first issue of Blackboard, the bulletin of the MTA (I). The genesis of MTA (I) itself has a curious history. Last year, it was decided by the mathematics olympiad cell to centrally administer a pre-regional mathematical olympiad exam in order to address several problems of non-uniformity. The competition was held last year with the assistance of the Indian Association of Physics Teachers (IAPT). The idea of forming a pan-Indian mathematics teachers association was mooted then with one of the aims being to assist in conducting the mathematical olympiad exams. However, the purpose of such an association was envisaged to encompass a much wider canvas and address all aspects of mathematics teaching in India. Subsequently, the MTA (I) was formed which includes school teachers as well as college teachers. A natural fall-out was the decision to bring out a bulletin of the MTA (I).

The Blackboard aims to be inclusive of mathematics teachers from all over India. Each issue is expected to carry articles related to high school and undergraduate mathematics. The issues will describe the works of Indian mathematicians, have articles outlining briefly current mathematical developments, and also address historical aspects of mathematics. There will be puzzles and problems for all, including high school level mathematics. All mathematics teachers (teaching in schools or colleges) will be encouraged to send in articles based on their classroom experiences and other aspects of interest to teachers.

The Blackboard will appear as an e-copy every three months. The first issue is proposed to be released during the inaugural MTA (I) conference during January 3-5, 2019. The International Congress of Mathematicians was held this year in Brazil and the works of Fields medalists and other awardees are described in this inaugural issue. The endeavour is to keep the style and level of coverage of most articles suitable and of interest to mathematics teachers as well as to students. The first issue carries a description of the research work of each ICM awardee and, this necessitates a somewhat higher level of exposition. However, there is a lot of material hopefully of interest to all. In future issues, several articles on the history of Indian mathematics, proofs of famous theorems proved by Indians are planned among other things. The World meeting of Women in Mathematics (WM^2) is an endeavour which will be described by Geetha Venkataraman in the next issue. This will give a social perspective as well.

R Ramanujam, who has been involved with mathematics education in the country for several decades has written about the importance of a problem solving culture. While seeking ways to make learning of mathematics enjoyable, a human aspect is to bring in history; the article by Amber Habib describes such an experiment in the classroom through a discussion of the sine function. The history of the mathematical olympiad programme in India is traced in an article by C R Pranesachar. Experts in the respective areas have described in reasonably simple language the works of all ICM medalists. This will really be of interest to students also. J K Verma has written a beautiful exposition on the theory of lattice points on polytopes which has connections with several areas of mathematics. In particular, there is an exciting Indian contribution related to this topic by a 50-year old work of Anand, Dumir and Gupta on counting magic squares. Anupam Saikia has reviewed a wonderful book by Davenport which has inspired and will doubtless continue to inspire future students to take up the study of mathematics as a career. There are puzzles and problems and crosswords which are of diverse levels; about half of them are accessible to students also. I have picked out a few problems posed by Ramanujan in Journal of the Indian Mathematical Society and conducted a brief discussion. The question of what is considered beautiful in mathematics, although subjective, has several candidates which are favourites of a large number of people. An article mentions a few of these and it would be interesting to receive the views of readers for future issues. The cover design will change from issue to issue. The work of the Fields medalists did necessitate a rather technical coverage this issue. But, the editors will strive to make the next issue onwards more balanced. The inaugural conference will find coverage in the next issue which is expected to carry a large share of articles accessible to school and college teachers. In fact, the teachers' feedback is invited and will be invaluable in bringing out the issues in a way that would benefit all levels of teachers and students. The readers are invited to find explanations for the various identities mentioned on the cover page - two of the problems address these.

... **B Sury, Indian Statistical Institute Bangalore.**

2 Message from President, MTA(I)

There is an abundance of evidence, anecdotal as well as statistical, showing that the mathematical education in the country is in dire need of reform at a fundamental level. Apart from interventions from the "top", the situation calls for a groundswell of awareness and a movement by the broad community in the country consisting of mathematics teachers and students at various levels, in grasping the issues involved and engaging with them, if the growing socio-cultural and intellectual needs of the society are to be met satisfactorily. The Mathematics Teachers' Association, founded in March 2018 (while its genesis was actually instigated by certain mundane factors recounted by B. Sury in his Editorial), has been acutely conscious of this broader context and has the objective of serving to promote a change through interactive participation of the community.

An inaugural conference bringing together many teachers who could discuss the underlying issues and brainstorm on remedial actions, and a publication of a Bulletin which would interface between MTA and the community have been the first two major initiatives in this respect. The inaugural conference will be held during January 3-5, 2019, at the Homi Bhabha Centre for Science Education (HBCSE, TIFR, Mumbai), and has received an enthusiastic response from the teaching community, including from a large number of high school teachers.

For bringing out a Bulletin an Editorial Board was constituted, in September 2018, with B. Sury as the Editor in Chief, and it is very heartening that the Board has put together within the short time-frame a nice set of articles, on *Blackboard*, that no doubt would enthuse many teachers and students of mathematics, in joining in and participating in the collective endeavour that is envisaged. In the process the Board has aptly availed, in particular, of the occasion of award of the Fields medals of 2018, a global event of great significance to the mathematical community, as a celebration of excellence in mathematics. On the other hand, as should be clear from the Editorial, *Blackboard* is committed to promotion of excellence as well as *antyodaya*, enlightening everyone, and it would surely be reflected in the forthcoming issues more transparently, as we move on, with greater involvement with teachers and students through their participation and feedback.

It is a matter of great pleasure, coupled with fond anticipation, that the first issue of *Blackboard* is being released, at our inaugural event, on January 3, 2019, the 188th birth anniversary of Savitribai Phule, who was instrumental in ushering major educational transformation in her time. The readers are urged to enjoy and engage with the ideas, and to participate in the activity in various ways. Be assured that your contributions and suggestions would receive a warm welcome, as *Blackboard* looks forward to serving as a discussion forum for the mathematical community.

... **S G Dani, UM-DAE Centre for Excellence in Basic Sciences (CBS), Mumbai.**

3 Needed: a problem solving culture *by* R Ramanujam

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> What does mathematics **really** consist of? Axioms (such as the parallel postulate)? Theorems (such as the fundamental theorem of algebra)? Proofs (such as Goedel's proof of undecidability)? Definitions (such as the Menger definition of dimension)? Theories (such as category theory)? Formulas (such as Cauchy's integral formula)? Methods (such as the method of successive approximations)?
>
> Mathematics could surely not exist without these ingredients; they are all essential. It is nevertheless a tenable point of view that none of them is at the heart of the subject, that the mathematician's main reason for existence is to solve problems, and that, therefore, what mathematics **really** consists of is problems and solutions.
>
> Paul Halmos

What Halmos says would resonate with anyone who enjoys mathematics. For most, enjoyment of mathematics begins with problem solving, and only later does it translate to the aesthetics and elegance of the other ingredients referred to by Halmos. For many outside the academia, solving a Sudoku puzzle offered by the newspaper might be a source of humble enjoyment. For those who travel the high roads of mathematical research, esoteric problems incommunicable to others might be their obsessions. For students of mathematics, or more generally, the mathematical sciences, problem solving is akin to daily physical exercise: without a daily regimen, their learning would not be "in shape".

All this may seem rhetorical, but if you ask a class of children in Class 9 what problem solving means to them, it is easy to see the abyss of perception between such rhetoric and what children perceive. This is not much different when we get to older students in classes 10 to 12, and rather sadly, with many undergraduate students as well. For most, problem solving is equated with *end of chapter exercises in textbooks*, no caveats whatsoever. The sole reason to solve these problems / exercises is to be able to do similar ones in examinations. Problem solving is a particular kind of questioning in tests peculiar to mathematics (and physics, to a lesser extent in chemistry, economics and a few other subjects). This has been confirmed to me by several batches of students over two decades. (Children who participate in Olympiads are a different breed altogether; I do not mean them in this discussion.)

When the same question is asked to school teachers of mathematics, the importance of problem solving is emphasized by most, and indeed glorified. However, when pressed for examples of problem solving experience, most revert to end of chapter exercises in textbooks. The experience with teachers of college mathematics is not very different.

The reason for this is obvious to all of us: the shadow of board examinations looms large over secondary education and influences every aspect of school, and in mathematics it translates to a particular style of questions asked in examinations which gets equated with problem solving. Since, more often than not, textbooks are written with preparation for examinations in mind, chapters develop material to "equip" students accordingly, and end of chapter exercises test the ability to answer similar questions. Those who set examinations refer to textbooks either created or prescribed by the Boards of education, and the cycle is complete.

Undergraduate education is less beset by this preponderance of examinations set far away, but by then, everyone is habituated to this style of testing. It is also a happy equilibrium when neither teachers nor students wish to deviate from the norm, rock the boat as it were.

(If I am specifically referring to secondary and tertiary education, it is not because problem solving is different in elementary schools. Class 7 September examination is no different from a Board examination in style. However, there seems to be some willingness to change at the primary and middle school stage, whereas later there is tremendous rigidity.)

Should it matter? It perhaps need not, had *enjoyment of mathematics* not become a casualty in all this. Even those who decry rote learning and calling for conceptual understanding to be tested in examinations miss this. Problem solving should be about every day classroom level enjoyment of mathematics, it should not be the exclusive domain of assessment of student's learning.

George Polya talks of various kinds of problems, and one of his categories is *Direct application and drill*. This is to emphasize the fact that working out a variety of problems directly stemming from definitions and theorems is essential for mathematics learning. This is needed to acclimatize oneself with textual material, and often this is needed for procedural fluency, without which one cannot address material coming up later. But then, these are only one kind of problems. Unfortunately, end of chapter exercises tend to be almost all of this kind, and school examinations (very kindly) follow their lead and most students miss problem solving experience of any other kind.

Open ended and *exploratory* problem solving and mathematical investigations are alien to most classrooms. Rather interestingly, most teachers say there is little time for this, as the syllabus leaves no room for 'such luxury'. Clearly, mathematical exploration is not seen as curricular activity. Many teachers are themselves unused to carrying out such exploration and even those acutely self-aware confess to having very few examples of such exploration at hand.

When asked for *Motivation and fore-runner problems*, those one would / should pose before starting a topic, to motivate the definitions coming up, many teachers express surprise at such a possibility. Perhaps this is natural in a classroom culture where one never questions definitions, and motivation is equated at best with "real life" applications.

What are problems that lead to enjoyment of mathematics at different stages of learning? Is it possible to construct problems that everyone can solve and yet offer variants that lead to challenges for the persistent? Can we have problems that start with hands-on activities and constructions (perhaps based on trial and error) that lead up the ladder of abstraction into esoteric conjectures and proofs? Can we distinguish problems that need clever tricks from those that demand creativity?[1]

All this is of course within the realm of the possible, and many creative teachers of mathematics have been doing this for a long time, offering the taste of mathematics to generations of students. But these are the stuff of individual heroic stories while the mainstream classrooms resemble physical drills where all children go through identical motions at the blow of a whistle.

Perhaps what is most urgently needed is creating a *healthy predisposition to problem solving* in our classrooms. I always say that when confronted by a mathematics problem that looks strange and unfamiliar, my first reaction is *PANIC*. I think this is normal, or at least I hope so. However, the point

[1]Of course the answer is yes, why else would I ask? This list cries out for examples of such problems. I desist from providing them now, but hope that the MTA-I becomes a forum for sharing them.

is to go on, keep at it, think a bit, recall 'stuff', try things. These are all delightfully vague, but actually help. We need to communicate to our students that it is OK to be daunted by the unfamiliar, but that when we persist, when we can make connections across many different themes, we make progress and there is enjoyment ahead. This is perhaps best done as a social activity, when students try things together, discuss, help each other, and see for themselves that different students bring differing strengths to situations. This is where exploratory problem solving is at its best, when there is something for everyone.

I propose one minimum standard for our classrooms. Can we ensure that every student engages in **one** enjoyable, exploratory mathematical activity every year of her school / college? For those who study mathematics for 10 years, this should mean at least 10 such experiences, more for those who go on with mathematics for higher levels. I hope I do not sound officious or insulting in offering such a low threshold, but a vast literature suggests that for a majority of students, enjoyable mathematics stops at the primary school, so it is indeed justified to propose such a minimum standard. On the other hand, if we cannot ensure even one exploratory mathematical activity per year for each student, what would "covering" the syllabus mean?

Succeeding in this requires ushering in a culture of problem solving to our classrooms, one where it is normal for students to talk mathematics. When every teacher carries in her notebook 10 problems to pose to students, preferably 5 of which she cannot solve, and the students in turn have problems for her (and for each other) to solve, we would all see a transformation of mathematics education, one that is meaningful and enjoyable.

4 History in the Classroom: Approximating the Sine Function *by* Amber Habib

Department of Mathematics, Shiv Nadar University.

When we seek ways to make the learning of mathematics more enjoyable, one suggestion is to increase the human element by bringing in the history of the subject. For example, the 2006 position paper of the National Focus Group for the Teaching of Mathematics stated that

> Lives of mathematicians and stories of mathematical insights are not only endearing, they can also be inspiring... Mathematics has been an important part of Indian history and culture, and students can be greatly inspired by understanding the seminal contributions made by Indian mathematicians in early periods of history.

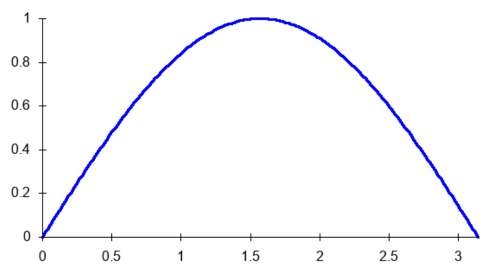
As time passes we view mathematics in different ways. To us, the solving of a quadratic equation is an exercise in manipulating symbols. Four thousand years ago it was part of the understanding of the areas of squares and rectangles. By taking up cases where the perspective has changed, we can gain insights into the connections between parts of mathematics. Since we know so little about the lives of the mathematicians of ancient and medieval India, this is the only way we can meaningfully bring history into the mathematics classroom.

In this note we shall describe an intriguing approximation of the Sine function that was given by Bhaskara, a mathematician who lived in the 7th century. He is often referred to as Bhaskara I to distinguish him from another, more famous, Bhaskara who lived in the 12th century. Bhaskara I's original work is contained in his 'major' and 'minor' works called the *Maha-bhaskariya* and *Laghu-bhaskariya* respectively. He is also recognized for his commentary on the work of Aryabhata. We will use his approximation as an opportunity to appreciate the usefulness of graphical descriptions of functions.

A Rational Approximation of the Sine Function

Prior to Bhaskara I, the Sine function was understood via tables of its values or of changes in its values. Perhaps these tables served to give the same insights that a graph gives us today. Even today, when we first learn to plot graphs we usually first make a table of values, so that the graph does not really capture new information but gives a different representation of the existing information. We shall see how useful this visual representation can be. Of course, we are not claiming that Bhaskara thought in this pictorial manner.

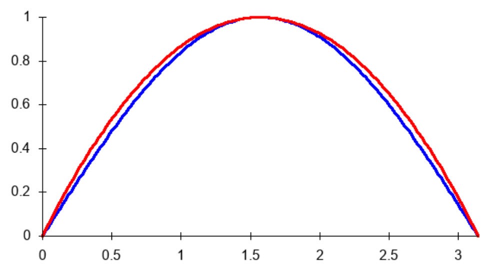
Let's start by looking at the $\sin(x)$ function from 0 to π, measuring angles in radians. The graph looks like part of an inverted parabola, the graph of a quadratic function:



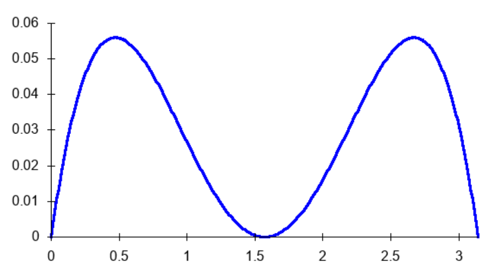
A quadratic function that is 0 at origin and π must have the form $y = Cx(\pi - x)$, for some constant C. Now we want the central value of y, at $x = \pi/2$, to be 1. Substituting $x = \pi/2$ and $y = 1$ gives $C\pi^2/4 = 1$ and we solve this to obtain $C = 4/\pi^2$. Thus we have a first approximation

$$\sin(x) \approx \frac{4x(\pi - x)}{\pi^2}$$

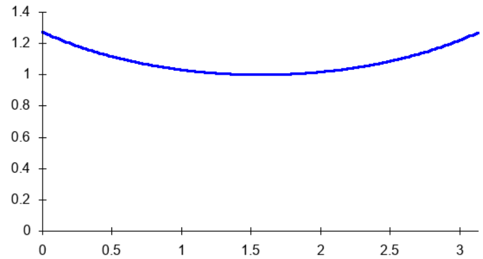
How good is this? Let's compare the graphs of the two functions:



Not bad! We could easily be satisfied with this. But Bhaskara was not, so let's take a closer look. There are two ways of testing the closeness of quantities: their difference could be close to zero, or their ratio could be close to 1. (This corresponds to whether we care about absolute error or relative error) Correspondingly, there are two ways of adjusting a quantity so that it becomes closer to another – by shifting or scaling. Let's first look at the difference $\dfrac{4x(\pi - x)}{\pi^2} - \sin(x)$:



This kind of shape can be generated by a 4th degree polynomial. But adjusting the coefficients of that polynomial so that it has zeroes and peaks at the right locations calls for quite a bit of fiddling. So let's look at the ratio of $\dfrac{4x(\pi - x)}{\pi^2}$ to $\sin(x)$ instead:



This looks much simpler – roughly a quadratic again! To match this, we need a quadratic that is 1 in the middle. We can't match the values at 0 and π as the ratio is undefined there. But we observe that the ratio is symmetric about $\pi/2$ and its value at $x = \pi/6$ is

$$\frac{4}{\pi^2} \times \frac{\pi}{6} \times \frac{5\pi}{6} \times 2 = \frac{10}{9}$$

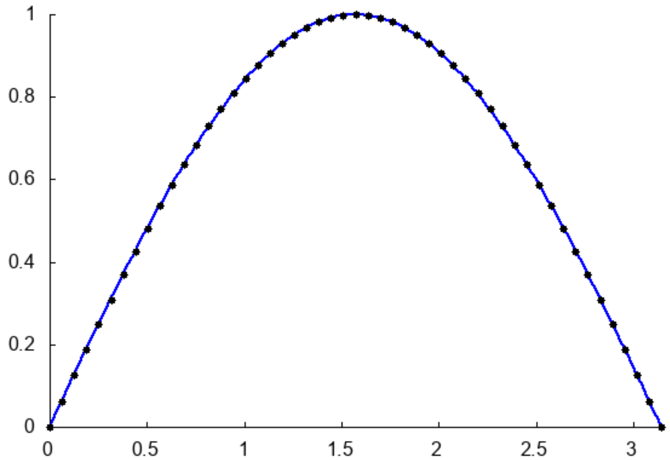
The symmetry tells us the quadratic should have the form $g(x) = A(x - \pi/2)^2 + B$. We want $g(\pi/2) = 1$ and this fixes $B = 1$. Then we want $g(\pi/6) = 10/9$ and this gives $A = 1/\pi^2$. So we have

$$\frac{4x(\pi - x)}{\pi^2 \sin(x)} \approx \frac{(x - \pi/2)^2}{\pi^2} + 1$$

or

$$\sin(x) \approx \frac{4\,x(\pi - x)}{(x - \pi/2)^2 + \pi^2} = \frac{16\,x(\pi - x)}{5\pi^2 - 4\,x(\pi - x)}.$$

This is precisely the approximation given by Bhaskara I if we translate it into modern notation! Let us ask, one last time, how good is our approximation? And answer again with a graph:



The two functions are virtually indistinguishable on this scale! Students who have learned calculus could try to estimate the maximum relative error. Others could plot the difference or ratio and look for the peaks.

What we have seen here is that a graph can give a good suggestion of the type of formula that can describe a function, and we can then use specific values of the function to refine the formula. This is the reverse of what we usually do, where we merely use data or a formula to plot the graph.

The article *The Bhaskara-Aryabhata Approximation to the Sine Function* by Shailesh Shirali (Mathematics Magazine, Vol. 84, No. 2, April 2011) goes into many aspects of this Sine approximation, such as its accuracy measured in different ways, how it compares to other approximations, and how the ideas it suggests can be used to develop fresh approximations. The history of the development of trigonometry in India offers several such episodes that can be used to enrich the classroom, including the motives for creating the Sine function, possible connections with Greek geometry, the techniques for tabulating Sine values, interpolation methods, and approximations by infinite series. A good source for much of this is the book *Mathematics in India* by Kim Plofker (Hindustan Book Agency, 2012).

5 Mathematical Olympiads in India *by* C R Pranesachar

Formerly of Mathematical Olympiad cell, HBCSE, TIFR at Department of Mathematics, Indian Institute of Science Bangalore.

Mathematical Olympiads are contests for gifted students. They are held at different levels, normally for individuals. Group contests also have been prevalent for a long time. Most of the contests are for younger students at the high school level and there are a few for undergraduate students also. These contests are now held worldwide and have their origins in the Hungarian 'Etovos' competitions which started in 1894. It took more than half a century to start International Mathematical Olympiads (IMOs) although the first IMO started with the small group of seven countries comprising the East European Bloc. The IMO started in 1959. Several other European countries such as England and France joined the race in 1960's and USA in 1970's. India's participation came much later, as the awareness of the competition was very limited.

In the mid-1980's Prof. J.N. Kapur of IIT Kanpur, a member of the National Board for Higher Mathematics (NBHM) persuaded the board members to start Indian National Mathematical Olmypiad (INMO) with the help of regional bodies. The interested candidates would first take the examination at the regional level in December, and the top 15 to 20 students from each region would be invited to write the national level Olympiad in February. About 300 to 400 students would participate in the INMO. The first INMO took place in 1986.

Earlier in the late 1960's, Prof. P.L.Bhatnagar of Indian Institute of Science (IISc) initiated Mathematical competitions, which were mainly held in Bangalore and surrounding cities in Karnataka. In the 1970's Chennai-based Association of Mathematics Teachers of India (AMTI) organized mathematical contests for Tamil Nadu (and some other states), and Andhra Pradesh Association of Mathematics Teachers started conducting contest in Andhra

Pradesh.

Mathematical Olympiads are written tests and the candidates have to solve 6 to 8 problems during a period of 3 to 4 hours. They are challenging, non-routine and require some ingenuity to get cracked. In the IMO the test is held on two consecutive days and on each day the contestant has to solve three problems in $4\frac{1}{2}$ hours. Each problem can fetch 7 points. Thus a student can score a maximum of 42 points. The medals are decided on cut-offs which vary from year to year. The topics in which the students have to be proficient are Algebra, Combinatorics, Geometry and Number Theory.

When India hosted the 37th IMO in Mumbai, 75 countries participated. This year in 2018 when the IMO was held in Romania, 107 countries participated. A student who scores 42 out of 42 has a 'perfect score'. The question papers are translated by the leaders of the accompanying teams into their National languages. Normally there are about 50 languages in which the problem set is translated. Although the answer scripts are evaluated by the leader and the deputy leader of the team, problem coordinators of the host country would also participate in the evaluation of all the scripts. There will be about 70 to 80 problem coordinators from the host country. The general rule is that nearly half the number of contestants will get some medal or the other, the Gold, Silver, Bronze medals being given in the ratio 1 : 2 : 3 to these toppers.

The problems are generally challenging and need a lot of ingenuity and talent to be solved. These are nonroutine problems not generally found in text books at the high school level. The problems are actually proposed by the participating countries and the host country will have a problem selection committee which sifts the problems and makes a shortlist of about 30-32 problems, nearly equally distributed over the four areas. The leaders of the country who assemble 3 to 4 days ahead of the students' arrival go through these short listed problems and vote for the final 6 problems in a democratic process. The tests, the evaluation process, the excursions and the medal distribution will take about ten to twelve days. Local hospitality will be taken care of by the host country. For the Indian team, the travel expenses are borne by MHRD. NBHM funds the local training camps at the RMO level, INMO level and for the IMO training camps. The IMO training camps are held for 4 weeks generally during April-May months every year. After a rigourous selection process six students are chosen to represent India in the

IMO held in July every year.

Prof. Izhar Hussain of Aligarh Muslim University initiated the process of participation of the Indian teams in the IMO's. Prof. Hussain took the responsibility of conducting RMO's and INMO's for several years until his untimely death in 1994. The first team was trained by only two resource persons over two years before being sent to represent India in the IMO in 1989. The later batches are being trained by 20 to 25 resource persons every year. The initial camps were held in IISc, Bangalore and BARC, Mumbai for the first few years. In 1996 the camps permanently shifted to Homi Bhabha Centre for Science Education (HBCSE), where training camps for Physics, Chemistry, Biology and Astronomy Olympiads are also held. So far India has bagged 11 gold medals, 64 silver medals, 66 bronze medals and 28 Honorable Mentions in its 30 appearances. There were 107 countries which participated in Romania in 2018. The highest number was 111 countries that participated in Brazil in 2017. Romania has held IMO five times. India's performance has not been up to the mark in the last 10 years or so except during 2011 and 2012. Also so far 27 problems proposed by India have been short listed and four of them have made it to the IMO final list.

When India hosted IMO in 1996, we gave away 35 gold medals, 66 silver medals, 99 bronze medals and 22 Honorable Mentions. The logo for the Indian IMO had the picture of a peacock and a snake taken from a problem from Lilavati written by Brahmagupta. In 1995, NBHM which is under Department of Automic Energy appointed under the chairmanship of Prof. M. S. Raghunathan, of Tata Institute of Fundamental Research, Mumbai, three members in the Mathematical Olympiad Cell. Prof. Phoolan Prasad took active role in the recruitment of the cell members. Prof. V.G. Tikekar who was the Chairman of the Mathematics Department of IISc provided office space for the cell. The cell members who were appointed in 1995 have retired and NBHM is looking for younger people to promote the Olympiad activity. Now the Olympiad Cell is located in HBCSE, Mumbai, and there is just one member looking after the olympiad activity.

We mention two important developments in recent times. Since 2015, India has started participating in the European Girls' Mathematics Olympiad (EGMO) and the Asia-Pacific Math Olympiad (APMO) as a Guest Nation. In general, the olympiad programme has taken positive initiatives in promoting girl students' participation in the olympiad activity.

Now we mention a few names who have been involved themselves in the Olympiad activity since its inception in 1989.

National Coordinators The following were National Coordinators for regional and National Olympiads: Prof. Izhar Husain, Prof. A.M. Vaidya, Prof. Rajeeva Karandikar, Prof.C. Musili, Prof. S.S. Sane, Prof. V.M. Sholapurkar, Prof. B. Sury (current).

MO Cell Members: Prof. C.R. Pranesachar (1995-2013), Prof. B.J. Venkatachala (1995-2015), Prof. C.S. Yogananda (1995-2005), Dr. Prithwijit De (2010-to date).

Chairman of NBHM: Prof. M.S. Narasimhan, Prof. M.S. Raghunathan, Prof. S.G. Dani, Prof. R. Balasubramanian, Prof. V. Srinivas (current).

Problems Proposed to the IMO's: In all 27 problems proposed by India have been short listed in the IMO's and 4 of them have made it to the final. These are

- C.R. Pranesachar, one problem, IMO 1990
- B.J. Venkatachala, two problems, IMO 1992, IMO 2002
- R.B. Bapat, one problem, IMO 1998

We also have a Hall of Fame of some Indian Medalists and INMO awardees:

- Subhash A. Khot, a two-time medalist was awarded the Waterman Award (2010), Rolf Nevanlinna Prize (2014), Fellow of the Royal Society (2017)
- Niraj Kayal and Nitin Saxena cracked 'the Primes are in P' problem in 2002
- Kannan Soundararajan was awarded Salem Prize (2003), Ostrowski Prize (2011), Infosys Prize (2011) SASTRA Ramanujan Prize (2005), Morgan Prize (1995)
- Sucharit Sarkar was awarded the Clay Research Fellowship

The above list is not complete. One can look at the following websites for information regarding Olympiads:

- https://olympiads.hbcse.tifr.res.in/
- https://www.imo-official.org/results.aspx

6 The Fields Medal

The International Congress of Mathematicians is a meeting that takes place every four years. Two to four mathematicians under the age of 40 are awarded the Fields Medal during the ICM. This is regarded as one of the highest honours a mathematician can receive. It has been described as the "Nobel prize of mathematics" although there are several differences. For instance, it is awarded to mathematicians under the age of 40 and the award is given only once in four years at the ICM since 1950. The award comes with a prize money of 15,000 Canadian dollars. The Canadian mathematician J C Fields established the award, and also designed the medal itself.

The first Fields medalists in 1936 were Lars Ahlfors and Jesse Douglas. The main purpose is to recognize and support young mathematicians who have made major breakthrough contributions. In 2014, the Iranian mathematician Maryam Mirzakhani became the first woman Fields Medalist - she tragically passed away in 2017. Manjul Bhargava was the first Fields medalist of Indian origin. In all, sixty people have been awarded the Fields Medal.

The most recent group of Fields Medalists received their awards on 1st August 2018 at the opening ceremony of the ICM held in Rio de Janeiro, Brazil. Caucher Birkar's medal was stolen shortly after the event and the ICM presented Birkar with a replacement medal a few days later. In what follows, the contributions of the four Fields medalists are detailed.

Caucher Birkar, Alessio Figalli, Peter Scholze, Akshay Venkatesh

6.1 The Mathematics of Akshay Venkatesh *by* A Raghuram

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Introduction

The first thing that strikes you is the sheer scope, breadth and depth, that characterizes Akshay Venkatesh's mathematics. As you look a little closer, the next thing that strikes you is the length of his published papers. At this point the reader should skip to the references to see that his papers are indeed very long. His mathematics is so broad that any other person, and certainly this applies to me, may not be able to do justice to all of it. If one is forced to name a subject that Venkatesh works in, then one might say it's Number Theory. But he develops and uses techniques from different areas such as Ergodic Theory, Differential Geometry, Algebraic Geometry, Lie Theory, Representation Theory, etc. Here's a sampling of the diverse topics he has worked on:

1. statistical distributions of arithmetical objects ([3], [5], [4]);
2. subconvexity results for L-functions ([7], [10]);
3. harmonic analysis on p-adic symmetric spaces ([9]);
4. (co-)homology of arithmetic groups ([1], [2]);
5. derived algebras ([6]);
6. automorphic and motivic cohomology ([8]).

The spectrum of problems illuminated by Akshay Venkatesh's insights is truly impressive! For this short article, in what follows I adumbrate two such problems that Venkatesh has worked on: first is in the realms of analytic number theory and the second in differential geometry. In each section I will first explain the context, and then Venkatesh's contribution.

Subconvexity for L-functions

The topic is best motivated with the classical Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

defined as a function of a complex variable s, with the series absolutely converging for $\Re(s) > 1$. Riemann proved in a landmark paper in the 1850's that this function admits an analytic continuation to a meromorphic function on the entire complex plane and satisfies the functional equation $(2\pi)^{-s}\Gamma(\frac{s}{2})\zeta(s) = (2\pi)^{s-1}\Gamma(\frac{1-s}{2})\zeta(1-s)$. There are deep mysteries encoded in the behaviour of $\zeta(s)$ in the critical strip $0 < \Re(s) < 1$. Probably the most famous open problem in number theory is the Riemann Hypothesis (RH) which asserts that every zero of $\zeta(s)$ in this critical strip is in fact on the critical line $\Re(s) = \frac{1}{2}$. A weaker conjecture is the Lindelöf hypothesis (LH)–weaker because it is implied by the RH–is that $\zeta(\frac{1}{2} + it) = O(t^{\epsilon})$ for any $\epsilon > 0$. Knowing the behaviour of $\zeta(s)$ on the lines $\Re(s) = 1$ and $\Re(s) = 0$, together with standard 'convexity bounds' in complex analysis, gives the *convexity estimate:* $\zeta(\frac{1}{2} + it) = O(t^{1/2})$. Any improvement over the convexity estimate towards LH is called a *subconvexity* result for $\zeta(s)$. At the bare minimum, a subconvexity estimate may be construed as lending evidence towards LH, and hence towards RH. The Riemann hypothesis being unreachable, breaking the convexity bound itself is a holy-grail as it often gives new number-theoretic insights. More generally, in the Langlands program, there are families of L-functions attached to automorphic representations of reductive groups that are the building blocks for the bridges between different areas of mathematics; the Riemann zeta function is literally the simplest example of an L-function in the Langlands program. A huge industry in modern analytic number theory is to prove subconvexity results for these L-functions.

Hitherto, most cases of subconvexity involved studying (and bounding) moments of the L-function at hand. Venkatesh, in a brilliant 109-page paper [10] in the Annals of Mathematics, which to date is his most cited paper, introduced a novel ergodic theoretic and geometric technique on bounding the size of periods of automorphic forms. To quote from the abstract of [10]:

> *The key features of this method are the systematic use of equidistribution results in place of mean value theorems ... .*

His method not only gives new subconvexity estimates for Jacquet–Langlands L-functions on GL(2), Rankin–Selberg L-functions on $\mathrm{GL}(2) \times \mathrm{GL}(2)$ and triple product L-functions on $\mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$, but also apparently admits generalization to higher groups, and furthermore, gives new bounds for Fourier coefficients of automorphic forms on these groups. For the purposes of this article, let me quote the simplest of his theorems on subconvexity (see Thm. 5.1 and (5.4) of [10]): Let π be an irreducible cuspidal automorphic representation of PGL(2) over a number field F of conductor $\mathfrak{p}$. Then

$$L(1/2, \pi) \ll_{\pi_\infty} N(\mathfrak{p})^{\frac{1}{4}-\frac{1}{2400}}.$$

The convexity estimate would have 1/4 in the exponent on the right hand side.

(Co-)homology of arithmetic groups

Let M be a smooth manifold. Some of the basic invariants attached to M are the homology groups $H_i(M, \mathbb{Z})$. Dually, we can also consider the cohomology groups $H^i(M, \mathbb{Z})$. Under finiteness hypothesis on M (which most reasonable manifolds satisfy) one knows that these groups are finitely-generated abelian groups, and hence have a free part and a torsion part. For the free part, we can tensor with a field, say $\mathbb{Q}$, and analyze the dimension of $H_i(M, \mathbb{Q})$ as a $\mathbb{Q}$-vector space (called the i-th Betti number of M) by various techniques. In contrast, the torsion part $H_i(M, \mathbb{Z})_{\mathrm{tor}}$, which is a finite abelian group, is far more mysterious. At the very least, one may ask for its order. The study of such homological invariants is especially interesting when $M = \Gamma\backslash G/K$ is a locally symmetric space; here G is a real-reductive group, K a maximal compact subgroup and Γ a discrete subgroup of G with finite co-volume. For a locally symmetric space, these invariants are related to automorphic forms on G and so to number theory involving the group G. The reader should keep in mind a simple example like $G = SL_2(\mathbb{R})$, $K = \mathrm{SO}(2)$ and Γ a finite index subgroup of $SL_2(\mathbb{Z})$. Even in such examples, $H_i(M, \mathbb{Z})_{\mathrm{tor}}$ is difficult to understand. One then attempts to understand the behaviour of these invariants in the limit: we may have a sequence $\{M_n\}$ of manifolds, with $M_{n+1} \to M_n$ being a finite-cover; for example $M_n = \Gamma_n\backslash G/K$, with $\Gamma_{n+1} \subset \Gamma_n$ being a subgroup of finite index.

In this context, Venkatesh, in collaboration with Bergeron [1] and furthermore with Sengun [2], has made some profound contributions to our understanding of torsion in the homology of arithmetic groups. Let me quote a simple and beautiful theorem in [2] which goes like this: Let $M_n \to M_0$ be a sequence of congruence arithmetic 3-manifolds, with M_0 compact. Assume that $\mathrm{vol}(M_n)$ goes to infinity. Then, under a certain conjecture they enunciate in that paper, as $n \to \infty$, we have:

$$\frac{\log |H_1(M_n, \mathbb{Z})_{\mathrm{tor}}|}{\mathrm{vol}(M_n)} \;\to\; \frac{1}{6\pi}.$$

This means that the order of $H_1(M_n, \mathbb{Z})_{\mathrm{tor}}$ grows exponentially with respect to the volume of M_n, i.e., we have a huge supply of torsion classes in the first homology of compact arithmetic 3-manifolds. This opens up deep questions concerning the arithmetic significance of torsion classes. (Incidentally, Peter Scholze, another Fields medallist of 2018, has also studied torsion classes, albeit in specific examples, and with a completely different focus of wanting to attach Galois representations to these torsion classes.)

To conclude, I would like to say that Akshay Venkatesh's work sheds a tremendous amount of light on the arithmetic mysteries encoded in the geometry of locally symmetric spaces, and his work opens up new and exciting avenues of research.

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6.2 Peter Scholze's work in Arithmetic Geometry *by* Kannappan Sampath

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Peter Scholze is a young German mathematician, who has arguably revolutionized arithmetic geometry through his work in p-adic Hodge theory. In his doctoral thesis, he introduced a class of "fractal-like" spaces, called *perfectoid spaces*, into this subject enabling him to compare the geometry over p-adic fields with geometry in characteristic p. In this brief introduction to his work, we would like to present some context for his work and indicate some early applications of this theory. In the short span of time since Scholze's work, this theory has seen numerous applications and inspired further developments in arithmetic geometry.

Arithmetic geometry is a subject born out of the rich interplay between the arithmetic of solutions to families of polynomial equations and the geometry of their zero loci. The zero loci of a family of polynomial (with integer coefficients) is an *algebraic variety*. Modern arithmetic algebraic geometry connects together apparently disparate mathematical objects in very beautiful and often mysterious ways. Much still remains conjectural and are subject of intensive research. At the outset, these conjectures predict an interplay between the symmetry in algebraic varieties, their geometry, and the arithmetic encoded in a generating function that possess a large group of symmetries.

Let us begin by illustrating this circle of ideas by giving some examples.

Arithmetic symmetries in roots of polynomial equations

Recall that a polynomial in one variable is said to be monic if its leading coefficient is 1. The group of symmetries in the set of solutions to a monic

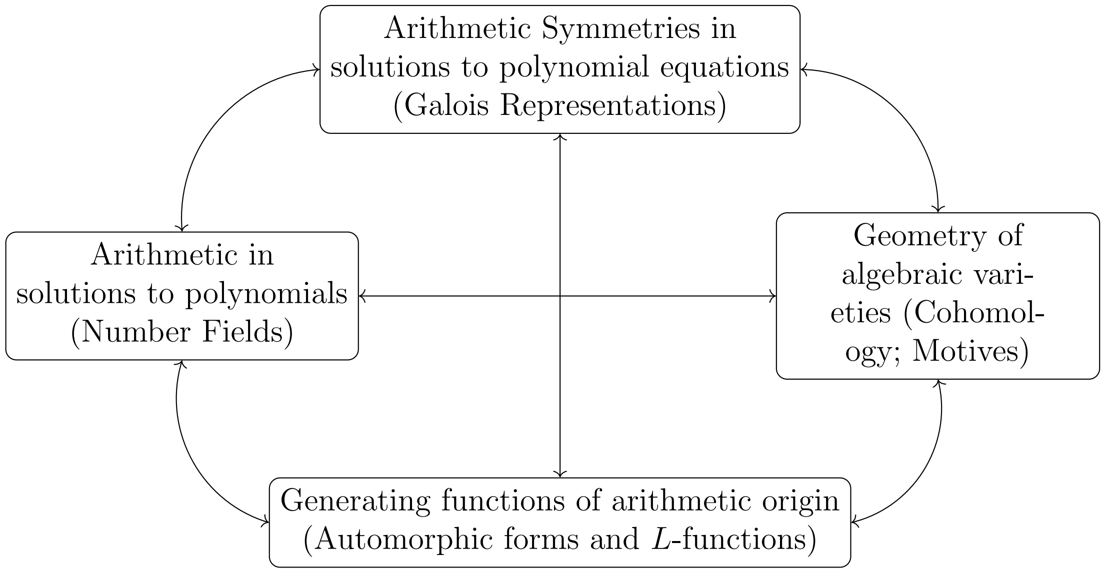


Figure 1: Bird's-eye view of Arithmetic Geometry

polynomial is called the *Galois group* of the polynomial. The group of all symmetries in solutions as we run over all possible monic polynomials is called the *absolute Galois group of the rational numbers*. This is the most fundamental object of interest in arithmetic geometry!
Example.

Let us take a monic quadratic polynomial with integer coefficients, say $x^2 - 2x + 3$. The set of solutions is then given by the quadratic formula

$$\{1 + i\sqrt{2}, 1 - i\sqrt{2}\}.$$

Note that the solution possesses a non-trivial symmetry: namely, the complex conjugate of a solution to this polynomial is still a solution to this polynomial.

As another example, let us take the polynomial $x^3 - 2$. The set of solutions is then given by $\{\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}\}$ where ω is a primitive cube root of unity. As in the last example, the complex conjugation is a symmetry; since $\sqrt[3]{2}$ is a real number, it is invariant under this symmetry; it is easily verified that the complex conjugate of ω is ω^2, so this symmetry interchanges $\omega\sqrt[3]{2}$ and

$\omega^2\sqrt[3]{2}$. There is another one in this case:

$$\sqrt[3]{2} \mapsto \omega\sqrt[3]{2}$$
$$\omega \mapsto \omega.$$

By arranging these roots as vertices of an equilateral triangle, one may visualize these symmetries. The complex conjugation corresponds to reflection about the vertex $\sqrt[3]{2}$ and the other symmetry corresponds to rotation counterclockwise. Thus, the Galois group of this polynomial coincides with the group of planar symmetries of an equilateral triangle.

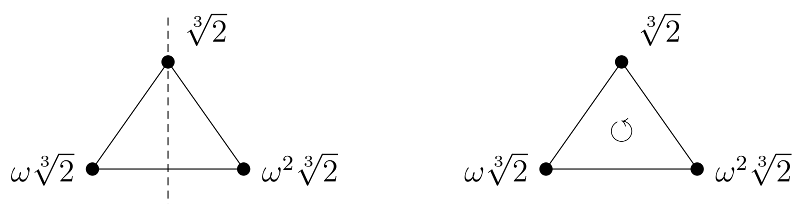
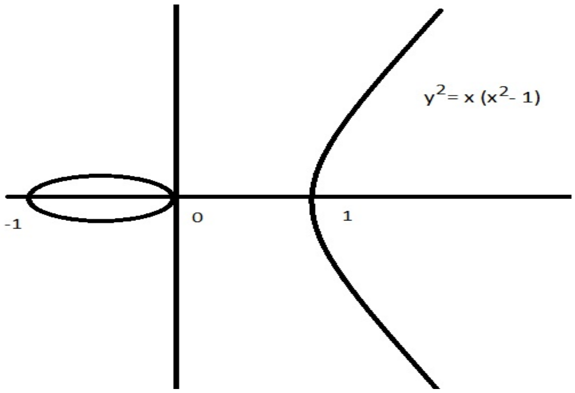


Figure 2: Symmetries of the solution to the polynomial $x^3 - 2$

The fundamental theorem of algebra tells us that any polynomial of degree n has n roots over the complex numbers. Thus the set of roots over the complex numbers is not a very interesting geometric object: it is a discrete collection of points. However, the situation changes dramatically, when one looks at polynomial equations in two variables.

Let us consider the polynomial equation $y^2 = x(x^2 - 1)$. The zero locus $\{(x, y) : x, y \in \mathbf{C} \text{ and } y^2 = x(x^2 - 1)\}$ over the complex numbers is called an elliptic curve; viewed over the complex numbers, the zero locus is, geometrically speaking, torus but this is not all obvious. And it follows from this that the zero locus has another surprising feature, namely that there is an addition law! That is, given two points P and Q satisfying the equation $y^2 = x(x^2 - 1)$, there is a third point $P + Q$ that also satisfies this equation.

To graph the set of complex solutions, namely $\{(x, y) \in \mathbf{C} \times \mathbf{C} : y^2 = x(x^2 - 1)\}$, we will need 2 complex dimensions or 4 real dimensions! So we content ourselves with a graph of real solutions to the equation $y^2 = x(x^2 - 1)$ which can indeed be drawn on the usual cartesian plane.



A rational point on the elliptic curve is a solution $P = (x, y)$ where both x and y satisfy a polynomial with integer coefficients. The arithmetic symmetry on the set of rational points is then realized by the absolute Galois group of $\mathbf{Q}$ permuting these rational points!

In general, the zero locus of a collection of polynomial equations does not carry all these additional structures. A theme in mathematics since the late 19th century has been to associate cohomology theories to geometric objects. A cohomology theory associates an invariant, typically a group, to a class of geometric objects and allows us to compare different geometric objects in that class. The symmetries in the geometry of these objects is then reflected in its cohomology.

Viewed as a complex manifold, every algebraic variety carries at least two different cohomology theories. The first, called the singular cohomology, is defined purely in terms of its topology. The second, called its de Rham cohomology, is an invariant defined using some analytic information on the algebraic variety; very loosely, it measures the extent to which the fundamental theorem of calculus fails on these algebraic varieties. In the 50s, efforts by many mathematicians culminated in a comparison theorem between these two cohomology theories.

An unsatisfactory point in these theories is that it is very hard to recover the arithmetic symmetries in the algebraic variety by studying these cohomology theories. In the 60s, Grothendieck and his collaborators developed a theory, called the étale cohomology, that allows us to recover these arithmetic symmetries! Thus, the upshot is that one can typically associate to an algebraic variety its arithmetic symmetries. This manifests itself as the action of the absolute Galois group on a certain cohomology theory for algebraic varieties.

Generating functions of arithmetic origin

We now turn to generating functions of arithmetic origin. A "generating function" is mathematicians' speak for a ledger or an accounting book. Except we are now going to keep track of some arithmetic invariant about algebraic varieties and in a power series in q.

To describe the arithmetic invariants we are going to keep track of, we must discuss a new system of arithmetic that is colloquially called the p-clock arithmetic.

The p-clock arithmetic

Let's briefly review an example: Suppose we start running a washer-and-drier system with your clothes in it at 11am. If the washer takes an hour, drier an hour and it takes an hour to fold clothes, then we should expect the laundry to be done at 2pm. In this case, notice how when we hit noon (12), we reset our numbers to 0 and start counting from 0.

This process can be done with any number instead of just 12. While we never multiply time, it turns out that the clock arithmetic system has a well-defined multiplication. In other words, just like the usual arithmetic, we can add two numbers, multiply two numbers, there is a number that behaves like 0 (in that adding this number to any other number gives that number back), there is a number that behaves like 1 (in that multiplying this number to any other number gives that number back). The clock arithmetic in which we reset our clocks at a prime number p is called p-clock arithmetic. In this case, it turns out that we might even divide a number by another non-zero number. Such a system of arithmetic with properties so close to the usual system of arithmetic (like rationals, real numbers or complex numbers) is called a *field*.

To illustrate this, let us consider the 3-clock arithmetic. In this arithmetic system, there are three distinct numbers $\{0, 1, 2\}$. And we have $1+2 = 3 = 0$, $2 \cdot 2 = 4 = 1$ and so on.

Equipped with this new system of arithmetic, we are now ready to describe our generating functions for the examples of last section.

Example.

Given a family of polynomials with integer coefficients, we may view them as polynomials with coefficients in p-clock arithmetic for every p. Instead of considering their complex solutions, we might want to count the number of solutions in p-clock arithmetic. For a polynomial f (as in examples of last section), let $N_p(f)$ denote the number of solutions to the polynomial equation $f = 0$ in p-clock arithmetic.

Let us take the polynomial $x^2 - 2x + 3$. Some of our intuition about polynomials with integer coefficients carries over even when working with p-clock arithmetic. For example, one easily checks by completing the square that this polynomial has 2 roots if and only if -2 is a square in p-clock arithmetic; and it has no solutions otherwise.

In the 2-clock arithmetic, $2 = 0$ and $3 = 1$, and so $x^2 - 2x + 3 = x^2 + 1$. So the polynomial has 2 solutions in 2-clock arithmetic, namely 1 repeated twice. In the 3-clock arithmetic, $3 = 0$ and so $x^2-2x+3 = x^2-2x = x(x-2)$. So the polynomial has 2 solutions in 3-clock arithmetic, namely 0 and 2. Here is a table of the number of solutions to this polynomial in p-clock arithmetic for primes up to 50:

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
$N_p(x^2 - 2x + 3)$	2	2	0	0	2	0	2	2	0	0	0	0	2	2	0

It turns out remarkably that $N_p(x^2 - 2x + 3)$ is the coefficient of q^p in the following q-series

$$\frac{1}{2} \sum_{(x,y)\in\mathbf{Z}^2} q^{x^2+2y^2};$$

in this sense, it is the generating function for the arithmetic data tabulated above. This q-series is called an 'Eisenstein series' of weight 1 and level 8.

Similarly, we have the following table for the example $x^3 - 2$:

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
$N_p(x^3 - 2)$	3	3	1	0	1	0	1	0	1	1	3	0	1	3	1

Once again, the number $N_p(x^3 - 2)$ is the coefficient of q^p in the following q-series

$$\sum_{(x,y)\in\mathbf{Z}^2} (-1)^{x+y} q^{9(x^2+3y^2)+3(x+y)+1}$$

This q-series is called a 'cusp form' of weight 1 and level 108.

For our final example, we return to the elliptic curve $y^2 = x(x^2 - 1)$.

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
$N_p(y^2 - x(x^2-1))$	2	4	8	8	12	8	16	20	24	40	32	40	32	44	48
$a_p = p + 1 - N_p$	1	0	-2	0	0	6	2	0	0	-10	0	-2	10	0	0

And yet again, for primes $p > 2$, the number a_p is the coefficient of q^p in the following q-series

$$q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} - q^{25} - 10q^{29} - 2q^{37} + 10q^{41} + 6q^{45} - 7q^{49} + \dots.$$

This q-series is a cusp form of weight 2 and level 32 and can be given an explicit closed form but it involves some subtle calculation in the Gaussian integers $\mathbf{Z}[i] = \{x + iy : x, y \in \mathbf{Z}\}$, namely those complex numbers whose real and imaginary parts are both integers.

The arithmetic generating functions that we have encountered so far are called "modular forms" (or more generally, automorphic forms).

Arithmetic and geometry over p-adic fields

Scholze studies algebraic varieties defined over p-adic fields. These fields have a fractal-like structure and so geometry over these fields could be quite counterintuitive. In this section, we shall briefly discuss what p-adic numbers are!

A p-adic number is an infinite series of the form

$$\sum_{k=-N}^{\infty} a_k p^k \tag{1}$$

where the $0 \leqslant a_k \leqslant p - 1$. The numbers $(a_{-N}, a_{-N+1}, \dots, a_0, a_1, \dots)$ are called the p-adic digits of this p-adic number. Just like decimal numbers, we will think of a_k as having place p^k. One can add two p-adic numbers by adding their p-adic digits with carry. Here is an example involving 3-adic numbers:

$$\left(\frac{1}{3^2} + \frac{1}{3} + 1 \cdot 3 + 2 \cdot 3^2\right) + \left(\frac{2}{3} + 1 \cdot 3 + 1 \cdot 3^2\right) = \frac{1}{3^2} + 1 + 2 \cdot 3 + 1 \cdot 3^3.$$

The reader should note how the digits with place value 3^{-1} when added result in a number with higher place value 3^0; similarly, the digits with place value 3^2 when added result in a number with higher place value 3^3. One can analogously multiply two p-adic numbers and one can divide a p-adic number by another non-zero p-adic number.

One of the first applications of the notion of a perfectoid space was to settle the weight-monodromy conjecture in many general cases. Suffice it to say that, this conjecture predicts a certain regularity in the cohomology of algebraic varieties over p-adic fields. The weight monodromy conjecture was proposed by Pierre Deligne in 1970; this conjecture was inspired by an analogous theorem he proved for algebraic varieties defined with p-clock arithmetic.

Enter: Scholze's perfectoid spaces

There is a very closely related field of Laurent series where the coefficients come from the field with p elements and the clock arithmetic. A Laurent series with p-clock arithmetic is an infinite series of the form

$$\sum_{k=-N}^{\infty} a_k t^k \tag{2}$$

where a_k are now between 0 and $p-1$. The numbers $a_{-N}, a_{-N+1}, \ldots, a_0, a_1, \ldots$ are called coefficients of the Laurent series. Despite the fact that expression (2) looks almost identical to (1) with t in place of p, there is one key difference between a p-adic number and a Laurent series: one uses the p-clock arithmetic where $p = 0$, $p + 1 = 1$ and so on. One can add two Laurent series with clock arithmetic. To illustrate this, let us consider the example of 3-adic numbers we had seen before, but now with 3-clock arithmetic:

$$\left(\frac{1}{t^2} + \frac{1}{t} + 1 \cdot t + 2 \cdot t^2\right) + \left(\frac{2}{t} + 1 \cdot t + 1 \cdot t^2\right) = \frac{1}{t^2} + 2 \cdot t.$$

The reader should note that $\frac{1}{t} + \frac{2}{t} = \frac{3}{t} = 0$; similarly, $2t^2 + t^2 = 3t^2 = 0$.

Two French mathematicians, J.-M. Fontaine and J.-P. Wintenberger, proved a remarkable theorem relating the group of symmetries of a polynomial $p(x)$ over the field of Laurent series with that of the polynomial $p^\sharp(x)$ obtained by replacing t with p^{1/p^n} where n is some large positive integer.

Scholze's ideas systematizes this construction allowing us to compare the cohomology of algebraic varieties over p-adic fields with cohomology of the algebraic varieties over fields with p-clock arithmetic. This allows Scholze to relate the weight monodromy conjecture of Deligne to Deligne's theorem for varieties defined with p-clock arithmetic!

We end here by remarking that an interested reader will find an introduction to these ideas in the expository articles by Scholze [4], Bhatt [1] and Fontaine [2]. A more advanced reader would enjoy the original paper [5] by Scholze introducing these objects.

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6.3 On the work of Alessio Figalli *by* Agnid Banerjee

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As is well known, Professor Alessio Figalli was awarded the Fields medal this year for his remarkable contributions in the field of Analysis, Partial Differential equations and Geometry. This is indeed a great incentive for the subjects of Analysis and PDE in which he is certainly one of the luminaries. Needless to say, a comprehensive overview of his scientific work is not possible in this short space given the fact that at an age of 34, he already has about 150 publications most of which are in leading mathematics journals. However more important than this number is the breadth and depth of his scientific articles which reflect his immense technical power, insights and creativity. Some of his works have genuinely opened up new directions for future research and this note is just a humble attempt to touch upon a few of them.

Figalli is one of the leaders in the use of optimal transport to various geometric and functional inequalities such as the isoperimetric inequality as well as its applications to regularity questions in certain nonlinear PDE's. Optimal transport roughly consists of finding the cheapest way of transporting a distribution of mass from one place to another. The first analysis of a prototypical optimal transport problem was carried out by G. Monge around 250 years ago. In the 1980's-1990's, several theoretical advancements were made in optimal transport which led to further applications in diverse areas such as town planning, image processing etc and it also triggered new developments within different areas of mathematics such as geometry and PDE. One of the most prominent example of this kind is the work of Figalli with F. Maggi and A. Pratelli on isoperimetric problems. The classical isoperimetric problem asks the question that among all smooth domains Ω with a prescribed surface area for the boundary $\partial\Omega$, which shape maximizes the volume and it turns out that the Euclidean balls are the unique extremizer of such an inequality. This in particular provides a quantitative justification for the precise spherical shape of soap bubbles which has to minimize a certain energy called the surface tension of the soap film. More generally while studying the physics of crystals which also attain a configuration that is energy minimizing and where the energy is dictated by the corresponding micro-structure, it is a natural question to ask as to how the shape of the crystal changes with an application of external energy. It turns out that this in turn can be cast as a problem concerning quantitative stability of certain anisotropic variants of the classical isoperimetric inequality. Figalli with his collaborators derived various sharp quantitative versions of such inequalities using optimal transportation which in particular establishes the stability of the configuration of crystals i.e. if the energy added is small, then the change in the configuration is moderate as well. Over here, we would also like to mention that isoperimetric inequalities are also connected to certain well known functional inequalities in analysis which are referred to as the Sobolev inequalities. This connection goes back to a visionary observation independently made by Mazya and Fleming-Rischell in 1960. Sobolev inequalities play a pivotal role in the existence and regularity for partial differential equations. In recent years, Figalli with his student R. Neumayer also studied similar stability questions for Sobolev inequalities and obtained a sharp quantitative result and thereby settling a long standing conjecture of Brezis and Lieb. Also in another remarkable joint work with Carlen, Figalli also addressed the stability analysis of logarithmic Hardy-Littlewood-Sobolev inequalities which was then used to describe the large time behavior for a certain evolution equation known as the critical mass Kuramoto-Sivashinsky equation that models diffusive instabilities in flame propagation type situations.

Another example of Figalli's work is his deep contribution to new regularity properties for the Monge Ampere equation which is a nonlinear second order equation of a special kind and arises naturally in several problems in Geometry such as the prescribed Gauss curvature and also in optimal transport. In a striking paper with De Philippis, he obtained a fundamental regularity result for solutions to the Monge-Ampere equation which in turn had applications to optimal transport maps. In particular, first with De Philippis, he showed that solutions to Monge Ampere equations with right hand side bounded away from zero and infinity have integrable second derivatives. Subsequently with O. Savin, using very novel and extremely original ideas, they showed a non-trivial higher integrability result for second derivatives of solutions which was the precise improvement needed to apply the optimal transport theory to the existence and regularity for an important nonlinear equation in fluid dynamics known as the semi-geostrophic equation. This later work was carried out by De Philippis and Figalli together with Ambrosio and Colombo and they obtained a completely answer for the case of three dimensional convex domains. We note that in the context of semi-geostrophic equations, Monge Ampere equations expresses the optimal transport of one density to another where the cost is a multiple of the square of the distance travelled. Given a certain density (say that of water particles in fluids), the solution of the Monge Ampere equation provides the optimal transport map and the Monge Ampere operator acting on the solutions represent the volume density of the image. We also note that this higher integrability result of Figalli with coauthors for second derivatives of solutions is in some sense one of the most significant development in the regularity theory for Monge Ampere equations since the ground-breaking work of Caffarelli in the early 1990's where he showed that for appropriately defined "generalized" solutions, if the right hand side is Hölder continuous, then all the second derivatives are in fact Hölder continuous.

Figalli has also obtained interesting regularity results for several nonlinear degenerate problems where the classical theory fails. The regularity theory for nonlinear elliptic operators with degenerate ellipticity at isolated points was pioneered by Uraltseva in the 1960's and such results were substantially streamlined and extended in the late 1970's and early 1980's in the fundamental works of Uhlenbeck, Lewis and Tolksdorf. Such equations arise in Non-Newtonian fluid flow. In 2013, in a joint work with M. Colombo, Figalli developed a very elegant regularity theory for elliptic equations whose degeneracy set can be a large convex set and obtained sharp results in this direction. A very interesting aspect of this work is that it surprisely combines perspectives from both divergence as well as non divergence theory.

We now turn our attention to Figalli's important contributions in the area of free boundary problems. A classical example of free boundary problem is the Stefan problem which consists of the studying the regularity property of the ice-water interface as time evolves when a block of ice is submerged in water. Another example is that of classical obstacle problem which consists of minimizing the standard "Dirichlet energy" among a class of functions with prescribed boundary values whose graphs stay above an obstacle. The graph of the solution to the obstacle problem can be thought of as a membrane with minimal energy attached to a fixed wire and which stays above the obstacle. In this problem, it turns out from simple examples that the membrane can actually touch the obstacle and a very fundamental question in this subject is to study the regularity of the boundary of the coincidence set. The ice-water interface in the Stefan problem or the coincidence set in the classical obstacle problem is referred to as the free boundary. In principle, free boundaries can have singularities like if one looks at the tip of icebergs or glaciers. A crowning achievement in the regularity of the free boundary for the classical obstacle problem was made by Caffarelli in 1977 where he showed that for a strictly concave obstacle, the free boundary is smooth and even real analytic (depending on the obstacle) outside of a set of singular points whose $n-1$ dimensional measure is 0. This essentially says that at most points, the free boundary is smooth. Later, Caffarelli also obtained a certain stratification result for singular points and showed that the singular points are contained in a C^1 submanifold. It turns out the study of regularity properties of free boundaries has a lot of resemblance with that of the regularity theory for minimal surfaces and this aspect is quite evident in the work of Caffarelli. Now very recently, Figalli together with Serra has obtained precise stratification result for the set of singular points in the classical obstacle problem and showed that the singular points are isolated upto dimension 3 and also obtained sharp results for the nature of singularities in higher dimension. In particular, they showed that the singular points are infact contained in submanifolds which are more regular than a C^1 submanifold and this significantly improves upon the previous results of Caffarelli. Subsequently in a joint work with Serra and Ros Oton, Figalli also obtained similar stratification result for the singular set in the free boundary for the Stefan problem. Other than that, in recent joint works with Shahgholian, Figalli has also extended the regularity theory of Caffarelli to certain non-trivial variants of the classical obstacle problem for fully nonlinear convex operators which in turn has provided a new perspective in the existing theory besides leading to further developments in the regularity theory for several nonlinear free boundary problems. There is another interesting variant of the classical obstacle problem known as the Signorini problem where the obstacle is situated at the boundary. This is an area which has picked up a lot in recent times since the breakthrough work of Athanasopoulos and Caffarelli in 2004 where they obtained optimal regularity of solutions to the Signorini problem. In a later work with Salsa, they obtained C^1 regularity of the free boundary (i.e. boundary of the coincidence set) at certain points on the free boundary where the solution separates from the free boundary at a minimal order. Such points are referred to as regular points. This analysis was further refined by Caffarelli, Salsa and Silvestre where they also related the classical obstacle problem for fractional laplacian to the signorini problem(or the thin obstacle problem) for certain degenerate elliptic operators.

It turns out that the complement of the regular set in the Signorini problem is significantly more complicated compared to the classical obstacle problem because there are examples to show that the solution can separate from the free boundary at different rates in the Signorini problem. This makes the situation quite different from that of classical obstacle problem where the solution always separate from the free boundary at a quadratic rate under strict concavity assumption on the obsctale. The first analysis of the singular set in the Signorini problem was carried out by Garofalo and Petrosyan using new geometric monotonicity formulas and they obtained a stratification result at singular points (which are precisely the points where the coincidence set is asymptotically negligible, say like a needle) and obtained a stratification result for the singular set similar to that of Caffarelli for the classical obstacle problem. However in the signorini problem, it turns out that the set of regular points (where the free boundary is nice) together with the set of singular points need not be all of the free boundary and there are explicit examples to show that the complement can indeed be quite large. However very recently, Figalli together with Barios and Ros-Oton showed that quite remarkably, under certain geometric assumptions on the obstacle, such a complementary set turns out to be empty and one has a similar structure theorem as that for classical obstacle problem. In a joint work with Caffarelli, Figalli has also made deep contributions in the classical obstacle problem for some evolutionary nonlocal equations which appear in Math finance.

Figalli has also made other impactful contributions in Analysis and Geometry which we now briefly mention. In recent times, a notion of nonlocal mean curvature came up in a seminal work of Caffareli, Roquejoffre and Savin in 2010. This allows to make sense of nonlocal minimal surfaces which corresponds to the case to vanishing nonlocal mean curvature. Since then, there has been a lot of activity in that subject. In this direction, in a very interesting work with Barrios and Valdinoci, Figalli showed that continuously differentiable nonlocal minimal surfaces are infact smooth. This combined with some previous work of Caffarelli and Valdinoci shows that nonlocal minimal surfaces are smooth upto dimension $N \leq 7$. In a joint work with Ciraolo, Maggi and Novaga, he also obtained a nonlocal version of the famous theorem of Alexandrov which says that constant mean curvature compact surfaces in the Euclidean space are spheres.

We finally would like to end this article with a quote of Prof. Luis Caffarelli about Figalli in his address at the 2018 International Congress of

Mathematicians in Brazil,

"Figalli's work is of the highest quality in terms of originality, innovation and impact both on mathematics per se as well as on its applications. He is clearly a driving force in today's global mathematical community"

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6.4 On the work of Caucher Birkar *by* Najmuddin Fakhruddin

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The work for which Caucher Birkar won a Fields Medal this year is in the field of algebraic geometry, more precisely in the birational classification of algebraic varieties. In this article, we will define algebraic varieties, explain what is meant by birational classification, and then briefly discuss some of Birkar's contributions.

Algebraic varieties are the zero sets of polynomial equations (in several variables), for example, the classical conic sections that we learn about in high school, i.e., ellipses, parabolas and hyperbolas. Elliptic curves, which are given by equations of the form $y^2 = x^3 + ax + b$, are more complicated varieties, and played a fundamental role in the Andrew Wiles's proof of Fermat's Last Theorem. More generally, we can consider zero sets of finitely many polynomials $f_1, f_2, \ldots, f_r$ in finitely many variables $x_1, x_2, \ldots, x_n$, i.e., we look at the set

$$V(f_1, f_2, \ldots, f_r) := \{(a_1, a_2, \ldots, a_n) \mid f_i(a_1, a_2, \ldots, a_n) = 0 \text{ for } i = 1, 2, \ldots, r\}\,.$$

Here the a_j could be elements of the field of real numbers $\mathbb{R}$ but it is actually better to allow them to be elements of the field of complex numbers $\mathbb{C}$, so $V(f_1, f_2, \ldots, f_r) \subset \mathbb{C}^n$. The reason for this is that the field $\mathbb{C}$ is algebraically closed, i.e., any nonconstant polynomial in one variable has a root in $\mathbb{C}$. Hilbert's Nullstellensatz, then says that any finite set $f_1, f_2, \ldots, f_r$ of polynomials as above (with coefficients in $\mathbb{C}$) has a common zero, i.e., $V(f_1, f_2, \ldots, f_r) \neq \emptyset$, iff for all other polynomials $g_1, g_2, \ldots, g_r$, $\sum_{i=1}^{r} g_i f_i \neq 1$. This is far from being true over the real numbers, e.g., consider the polynomial $x^2 + 1$. So even though complex numbers are more complicated than real numbers, (algebraic) geometry over $\mathbb{C}$ is easier than over $\mathbb{R}$!

In particular, $\mathbb{C}^n$ itself is an algebraic variety (it is the zero set of the polynomial $f \equiv 0$) and from now on we denote it by $\mathbb{A}^n$ (called n-dimensional *affine* space). The algebraic varieties that we have defined above in $\mathbb{A}^n$ are called *affine* varietes. It turns out that it is better to consider *projective* varieties. These are subsets of $\mathbb{P}^n$ (which is called *projective* n-space), which is defined to be $\mathbb{C}^{n+1}\backslash\{(0,\ldots,0)\}/\sim$, the equivalence classes for the equivalence relation given by $(a_0,\ldots,a_n) \sim \lambda(a_0,\ldots,a_n)$ where $0 \neq \lambda \in \mathbb{C}$. We suggest that the reader try to see that $\mathbb{P}^1$ is given by gluing two disjoint copies of $\mathbb{C}$ with a in the first copy identified with $b = 1/a$ in the second copy if $a \neq 0$. For projective varieties, instead of looking at zero sets of arbitrary polynomials we must consider only homogenous polynomials, i.e., polynomials such that each monomial in it is of some fixed degree d. The reason for this is that only the zero sets of such polynomials are preserved by the equivalence relation. For example, we can take f to be $x_0^2+x_1x_2+x_2^2$ but not $x_0^2+x_1+x_0x_2^2$. Any polynomial in n variables can be made into a homogenous polynomial in $n+1$ variables by multiplying all the monomials by appropriate powers of x_0; this allows us to get a projective variety from an affine variety. For example, $x_1^2+x_2^2-1$ becomes $x_1^2+x_2^2-x_0^2$, and x_1x_2-1 becomes $x_1x_2-x_0^2$. If we change variables by $x_1 \mapsto x_1-x_2$, $x_2 \mapsto x_1+x_2$, the second polynomial becomes $x_1^2-x_2^2-x_0^2$, so the hyperbola becomes a circle: in projective space all conics become "equal"! This illustrates one of the reasons for preferring projective varieties.

We can also get an affine variety from a projective variety, in fact several different ones, by setting one of the variables equal to 1 (so the resulting polynomials have one fewer variable and are not necessarily homogenous). A more general class of varieties, containing both affine and projective varieties, is that of *quasiprojective* varieties: these are of the form $X\backslash Y$ where X and Y are projective varieties.

Another reason for preferring projective varieties can be seen from the fact that any two distinct lines (i.e., varieties defined by one equation of degree 1) in $\mathbb{P}^2$ intersect in a unique point, so there are no parallel lines! From the point of view of topology, $\mathbb{A}^n$ is not compact but $\mathbb{P}^n$ (with the quotient topology) is, which also makes the latter nicer.

Even after replacing $\mathbb{R}$ by $\mathbb{C}$ and then $\mathbb{A}^n$ by $\mathbb{P}^n$, there are still too many varieties to admit a reasonable classification, so we need to impose some conditions on the varieties that we want to classify. We say that a variety V is *irreducible* if it cannot be written as a union of two proper subvarieties, i.e., if $V = V_1 \cup V_2$ for some other varieties V_1 and V_2 then $V = V_1$ or $V = V_2$. Any variety V can be written in a unique way as a (minimal) finite union of irreducible subvarieties—if V is defined by a single polynomial f then $V = \bigcup_i V(f_i)$, where the f_i are the distinct irreducible factors of f—so we try to classify only irreducible varieties; from now on, when we say variety we shall (usually) assume that it is also irreducible.

In order to classify mathematical objects, one usually attaches some invariants to them. For example, finite dimensional vector spaces over a field (e.g., $\mathbb{R}$ or $\mathbb{C}$) are classified by their dimension. Any variety V also has a dimension, denoted $\dim(V)$. We do not define this precisely, but if $V = \mathbb{A}^n$ or $\mathbb{P}^n$ then its dimension is n and if $V = V(f)$ (in $\mathbb{A}^n$ or $\mathbb{P}^n$) is defined by a single nonconstant polynomial f, then its dimension is $n-1$. The only 0 dimensional variety is a single point, but for all $n > 0$ there are infinitely many (non-isomorphic, i.e., not related by a "change of coordinates") varieties of dimension n.

Even classifying all one dimensional varieties is quite difficult, however the problem can be made simpler by restricting to *smooth* varieties. Again, we do not define what this means precisely, but intuitively it means that the variety has no "sharp" points or crossings. For example, a circle is smooth but the curves (one dimensional varieties) given by the polynomials x_1x_2 or $x_1^2 - x_2^3$ have "singularities" at $(0, 0)$. The Fermat hypersurfaces $V(f) \subset \mathbb{P}^n$ with $f = x_0^m + x_1^m + \cdots + x_n^m$ are smooth for all positive m and n. (More generally, if $V = V(f)$ is a projective variety given by a single polynomial f then it is smooth if and only if f and all its partial derivatives have no common zero except $(0, \ldots, 0)$.) If a variety is not smooth we say that it is *singular*.

A celebrated theorem of Hironaka (for which he received the Fields Medal in 1966) asserts that any projective variety V is *birational* to a smooth projective variety W. Here birational roughly means that after removing a proper subvariety V' from V and W' from W we get the same (isomorphic) varieties. If V is a curve then the theorem is easy and the smooth variety is unique, but if $\dim(V) > 1$ there are always infinitely many choices for W.

The classification of smooth projective curves has been essentially known since the time of Riemann. To each such curve one can attach another invariant called the genus which is a non-negative integer. For a smooth curve in $\mathbb{P}^2$ given by an equation f in x_0, x_1, x_2 this number is $(d-1)(d-2)/2$, where d is the degree of f; note that we do not get all positive integers this way. Nevertheless, the classification theorem for curves says that for any $g \geq 0$ there exists a curve of genus g, in fact the set of isomorphism classes of curves of genus g correspond in a natural way to the set of points of a variety M_g (called the moduli space of smooth projective curves of genus g) where

- M_0 is a single point (the only curve of genus 0 is $\mathbb{P}^1$),
- M_1 is $\mathbb{A}^1$ (corresponding to the "j-invariant" of an elliptic curve),
- M_g has dimension $3g - 3$ if $g > 1$.

To understand the classification in more detail one studies the properties of the varieties M_g, but we do not discuss this here.

To get some feeling for birational varieties, we now discuss the process of blowing up: given a smooth n-dimensional variety V and a point $v \in V$ this produces a new (smooth) variety $\mathrm{Bl}_v V$ which is almost the same as V except that v is replaced by $\mathbb{P}^{n-1}$. Since $\mathbb{P}^0$ is a single point we get nothing new if $n = 1$, but if $n > 1$ this gives a variety which is birational to V but not isomorphic to V. We can choose various points in V and also iterate this procedure, so if $\dim(V) > 1$ we get infinitely many (smooth, projective if V is projective) varieties birational to V. Let us consider the case when $V = \mathbb{A}^2$ and $v = (0,0)$ in more detail. Then $\mathrm{Bl}_0\mathbb{A}^2$ is $(\mathbb{A}^2 \coprod \mathbb{A}^2)/\sim$, so we are gluing two disjoint copies of $\mathbb{A}^2$ and the equivalence relation $\sim$ is given by identifying (a_1, a_2) in the first $\mathbb{A}^2$ with $(b_1, b_2) = (1/a_2, a_1a_2)$ in the second $\mathbb{A}^2$ if $a_2 \neq 0$; it can be shown that this is a quasiprojective variety. We get a map $p : \mathrm{Bl}_0\mathbb{A}^2 \to \mathbb{A}^2$ by sending (a_1, a_2) in the first $\mathbb{A}^2$ to (a_1, a_1a_2) and (b_1, b_2) in the second $\mathbb{A}^2$ to (b_1b_2, b_2). It is easy to see that $p^{-1}(v)$ is a singleton if $v \neq 0$, but $E = p^{-1}(0)$ (the so called "exceptional divisor") is the union of all $(0, a_2)$ in the first $\mathbb{A}^2$ and $(b_1, 0)$ in the second $\mathbb{A}^2$ which is exactly $\mathbb{P}^1$. The map p is called the blowup of $0 \in \mathbb{A}^2$ or the blow down of $E \subset \mathrm{Bl}_0\mathbb{A}^2$.

Having understood blowups we return to the question of classification. Since there are infinitely many varieties birational to a given variety V in general, we first try to find a *minimal model* of V. Roughly speaking, this is a variety W which is birational to V (so a "model" of V) and which is not obtained from blowing up any other variety (so it is "minimal"). It turns out that for smooth projective two dimensional varieties, i.e. *surfaces*, this can always be done and moreover the minimal model is usually, but not always unique. The varieties for which the minimal model is not unique are the so called uniruled varieties: these are those varieties V such that for all $v \in V$ there is a copy of $\mathbb{P}^1$ contained in V and containing v, e.g. $V = \mathbb{P}^n$. All this was understood close to a hundred years ago by the so called Italian school of algebraic geometery (Castelnuovo, Enriques, Severi,...).

Finding minimal models of varieties of dimension > 2 turned out to be significantly more difficult. A basic stumbling block was the fact that there need not exist any smooth minimal model! It was Mori in the 1980's who discovered the correct way to proceed and formulated the Mori program or the Minimal Model Program (MMP). One is forced to give up the luxury of dealing with only smooth varieties and some varieties with "mild" singularities have to be allowed. Mori then succeeded in constructing minimal models for three dimensional varieties and for this work he was awarded the Fields Medal in 1990.

In the MMP, one starts with a smooth projective variety V and tries to make it minimal by systematically blowing down (i.e., contracting) suitable subvarieties of V. At some stage the variety might become singular, but if the singularities are "mild" one can still continue. However, if $\dim(V) \geq 3$ it can happen that the singularities are very bad ("non $\mathbb{Q}$-factorial") and so one gets stuck. To get around this one has to introduce a new operation called a "flip" which tries to improve the singularities (without blowing up). The main difficulty with this is proving that this can always be done (existence) and that the program will stop after finitely many steps (termination).

For existence, Mori had to use very detailed classification of three dimensional singularities which was impossible to extend to higher dimensions. In the first major work of Birkar (in collaboration with Cascini, Hacon and McKernan) the existence of flips was proved in all dimensions using an inductive procedure and without explicit computations. Although termination is still not known in general, the above authors managed to prove it in many important cases, sufficient to imply the finite generation of the *canonical* ring (also independently proved by Siu).

This is a graded ring $R(V) = \bigoplus_{i=0}^{\infty} R_i$ attached to a smooth projective variety V (defined using "pluricanonical forms"), where R_0 is $\mathbb{C}$ and each R_i is a finite dimensional $\mathbb{C}$-vector space. Two varieties which are birational have the same canonical ring, so it is a birational invariant. If this ring is finitely generated, using it one can construct a variety V^{can}, called the canonical model of V; birational varieties have the same canonical model. Now $\dim(V^{can}) \leq \dim(V)$; if the dimensions are equal then V^{can} is actually birational to V and V is said to be of *general type*. For example, if $V = V(f)$ is a smooth subvariety of $\mathbb{P}^n$ then V is of general type if $\deg(f) > n+1$, so in some sense most varieties are of general type. However, it can happen that V^{can} is a point, or even the empty set, in which case we do not get any information about V from the canonical ring. In fact, in a suitable sense, any variety can be built from varieties of general type, varieties with V^{can} a point (Calabi–Yau varieties, e.g., $V(f)$ with $d = n+1$) and a class of varieties with $V^{can} = \emptyset$, the so called Fano varieties (e.g., $V(f)$ with $d \leq n$). We mention here that the *Abundance conjecture*, one of the fundamental conjectures of the MMP, which implies a precise form of the statement in the previous sentence, is still known only in dimensions ≤ 3.

While there are still many mysteries associated to Calabi–Yau varieties, the second major work of Birkar proves a fundamental result about Fano varieties, the so called BAB conjecture. Here it is crucial for applications that one allows certain singular varieties, the result for smooth varieties being known more than two decades ago. In essence, the result says that for any $\epsilon > 0$ there exists a function $F_\epsilon : \mathbb{N} \to \mathbb{N}$ such that all Fano varieties of a given dimension n and with bounded singularities ("$\epsilon - lc$"), can be embedded in (i.e., are isomorphic to subvarieties of) $\mathbb{P}^{F(n)}$. This is the first step in constructing moduli spaces for Fano varieties (like the spaces M_g for curves).

A striking application of this result is as follows: A theorem of Jordan from the nineteenth century says that there is a function $J : \mathbb{N} \to \mathbb{N}$ so that any finite subgroup of $GL_n(\mathbb{C})$ (the group of invertible $n \times n$ matrices) has an abelian normal subgroup of index $\leq J(n)$. The n dimensional Cremona group C_n is the group of automorphisms of the field $K_n = \mathbb{C}(x_1, x_2, \dots, x_n)$ (the fraction field of the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_n]$) which are the identity on $\mathbb{C} \subset K_n$. While the explicit structure of this group is known only when

$n = 1$, and it is known to be infinite dimensional for $n > 1$, as a consequence of the results of Birkar (via a theorem of Prokhorov and Shramov) it follows that there exists a function $B : \mathbb{N} \to \mathbb{N}$ so that any finite subgroup of C_n contains an abelian normal subgroup of index $\leq B(n)$, i.e., the Cremona groups have the Jordan property.

7 ICM 2018 Prizes - Chern, Gauss, Nevanlinna, Leelavati *by* Geetha Venkataraman

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Masaki Kashiwara: The Chern Medal 2018 awardee

The Chern Medal was awarded to Masaki Kashiwara for outstanding, foundational and sustained contributions over an almost 50 year period to algebraic analysis and representation theory. The ICM 2018 video [1] about the awardee shows a sprightly 71 year old Kashiwara leaping over stones, crossing a stream to go to his office in Kyoto, Japan. Masaki is an emeritus professor with Kyoto University. The Chern Medal is awarded once in four years during the ICM to an individual "whose accomplishments warrant the highest level of recognition for outstanding achievements in the field of mathematics. All living, natural persons, regardless of age or vocation, shall be eligible for the Medal" [2]. Kashiwara credits his start in Mathematics to his supervisor Mikio Sato. As a graduate student he learnt algebraic analysis and representation theory. He began developing microlocal analysis which uses Fourier transforms to the study of partial differential equations, localisation with respect to location of the point in the space as well as the cotangent space at that point. The concept of D-modules started by Sato and then completed by Masaki, allows a bridge between areas of algebra and analysis. A D-module is a module over a ring D of differential operators. Kashiwara used D-modules in 1980 [3] to prove the generalised Riemann-Hilbert Problem: for every monodromy group, is there an associated linear differential equation whose behaviour near a singularity is described by that group? A seminal contribution made by Kashiwara to representation theory and one which bears his name is the 'Kashiwara Crystal Basis'. This allows for the use of counting arguments or combinatorics to answer questions posed in representation theory. Both D-modules and Kashiwara Crystal Basis have become essential tools in many areas of mathematics. For a short, reader-friendly account of D-modules and Crystal Basis, see [4] and for a short mathematical talk about Masaki Kashiwara's work of fifty years see

[5]. Another interesting nugget of information from the ICM video, is that Kashiwara has several hundreds of notebooks filled with mathematics. He learnt to write his mathematics in notebooks after losing the sheets in which he recorded his ideas in his early days as a mathematician. The sheer number of these notebooks is a further affirmation of his prolific mathematical contributions at the highest level.

David Donoho: The Gauss Prize 2018 awardee

The Gauss prize is awarded to a scientist whose research in mathematics has had an impact outside of the mathematical realm be it in technology, in business, or simply in people's everyday lives. David Donoho's mathematical work ticks all the three boxes. David is a professor at Stanford University in the Department of Statistics, as well as the Anne T. and Robert M. Bass Professor in the Humanities and Sciences. David's plenary lecture at ICM 2018 [6] had several deeply personal anecdotes that brought to the fore the need for faster and better MRI sensing. One of the anecdotes was about the fact that his wife had undergone brain surgery when she was younger and that inspired his son to become a neurosurgeon. The need to know via MRI scanning as to what can be expected before opening up the brain of a patient is not just essential, it is one of the main tools in planning ahead for the surgery. Through his son and other medical fraternity, David discovered the problems that existed with MRI scanning in the early 2000's. Donoho also lost his father around then to an aggressive form of prostrate cancer which was not diagnosed properly. Another doctor who had been working on prostrate cancer mooted the idea of mass MRI scans for prostrate cancer as a diagnostic tool, a sort of 'Manogram' as David put it in his ICM lecture. 'Better, Faster MRI's' would not just save money in the longer run by delivering accurate diagnosis, say by catching an aggressive type of prostrate cancer at an early stage but most importantly save lives. What all these connections spurred David to do was to develop the mathematics by 2006 [7] which has now been used by three of the technology giants in creating the better, faster MRI. This is only one aspect of David's work but according to Emmanuel Candés, whom David supervised for his doctoral thesis and who delivered the Gauss Prize Laudatio [8] on David Donoho's mathematical work and its impact, David is a master at exploiting sparsity in order to increase efficiency. David's research interests, on the Stanford website reads simply as “My theoretical research interests have focused on the mathematics of statistical inference and on theoretical questions arising in applying harmonic analysis to various applied problems. My applied research interests have ranged from data visualization to various problems in scientific signal processing, image processing, and inverse problems.”

Constantinos Daskalakis: The Nevanlinna Prize 2018 awardee

The following is the citation for the Rolf Nevanlinna Prize [9] ’The Rolf Nevanlinna Prize is awarded once every 4 years at the International Congress of Mathematicians, for outstanding contributions in Mathematical Aspects of Information Sciences including: (i) All mathematical aspects of computer science, including complexity theory, logic of programming languages, analysis of algorithms, cryptography, computer vision, pattern recognition, information processing and modelling of intelligence. (ii) Scientific computing and numerical analysis. Computational aspects of optimization and control theory. Computer algebra. The Rolf Nevanlinna Prize Committee is chosen by the Executive Committee of the International Mathematical Union. A candidate’s 40th birthday must not occur before January 1st of the year of the Congress at which the Prize is awarded.’

Thirty seven year old Constantinos Daskalakis, a professor of computer science at MIT was awarded this highest honour accorded to computer scientists. An Indian connection is that Manindra Agarwal was part of the Nevalinna Prize Committee for 2018. Daskalakis was given the Nevanlinna prize for “transforming our understanding of the computational complexity of fundamental problems in markets, auctions, equilibria and other economic structures. His work provides both efficient algorithms and limits on what can be performed efficiently in these domains” [10]. Constantinos works in the interface of computer science and economics and is interested in using mathematics to understand humans. He showed that while Nash equilibrium exists in complex games, it would be computationally impossible to attain such equilibria, in the sense that it might take hundreds of years of computational time to find that equilibria [11]. This is done by showing that finding Nash equilibria is PPAD-complete. PPAD is a subclass of NP (solution is quickly checkable, that is, in polynomial time), and stands for ‘Polynomial Parity Arguments on Directed graphs’. PPAD was introduced by Daskalakis’s doctoral supervisor Christos H. Papadimitriou in 1994. While theoretical solutions exist for PPAD problems, finding them is computationally intractable. The proof that Nash is PPAD and PPAD-complete involved showing that Brouwer's fixed point theorem was also PPAD-complete. For a short but detailed account of Constantinos' work see [12]. We end this part with a few lines from the acknowledgements part of Constantinos's thesis [13]: "Christos once told me that I should think of my Ph.D. research as a walk through a field of exotic flowers. "You should not focus on the finish line, but enjoy the journey. And, in the end, you'll have pollen from all sorts of different flowers on your clothes." I want to thank Christos for guiding me through this journey and everyone else who contributed in making these past four years a wonderful experience."

Ali Nesin: The Leelavati Prize 2018 awardee

"The Leelavati Prize is intended to accord high recognition and great appreciation of the International Mathematical Union and Infosys of outstanding contributions for increasing public awareness of mathematics as an intellectual discipline and the crucial role it plays in diverse human endeavors" [14]. Started as a one-time award during the ICM 2010 which was held in Hyderabad, India, it was converted into a prize to be awarded once in four years at the ICM. The Leelavati Prize recognises Ali's "outstanding contributions towards increasing public awareness of mathematics in Turkey, in particular for his tireless work in creating the 'Mathematical Village' as an exceptional, peaceful place for education, research and the exploration of mathematics for anyone" [15]. "Truth is stranger than fiction' is an apt phrase when you see the path that Ali has traversed in his life. Ali's journey to developing and creating the Mathematical Village began after his father, Aziz Nesin, a renowned writer and socialist, died in 1995. Ali left the University of California, at Irvine, to return to Turkey to look after the foundation that his father had begun. As Ali says, America did not need him but Turkey did. He joined the Department of Mathematics at the Istanbul Bilgi University. He felt that Turkey lacked access to elite, quality education in Mathematics. Students coming to the University were ill-prepared to handle mathematics. Worse, they seemed scared of looking foolish to clear their doubts. He began summer schools to train students but that soon became difficult to sustain; where would one be able to host students each year? Whenever Ali and his Armenian friend, Sevan Nisanyan would get together they would dream of building a maths village where students could live life, learn and do mathematics. Sevan was an architect and was to design the village. Work began in 2007, in a village near Izmir, with very little money but many volunteers willing to put in the labour required. However, the construction was deemed illegal, and the police came to shut down the work and the summer school that was happening. The group then moved to a nearby forest where they were harassed again. At this point Nesin went public about the happenings related to the maths village and huge support rolled in from the public in terms of money and goodwill. The harassment was probably a result of Ali's father's political beliefs. Ali and his group of volunteers removed the seals placed by the police and began construction again, bringing to life Nisanyan's architectural design. In 2014, Nisanyan was jailed for illegal construction. He managed to escape in 2017 and lives in Greece. Nisanyan was with Ali during ICM 2018 in Rio, relating their adventures with pride at the award of the prize for their Mathematical Village. The Nesin Mathematical Village, nestled in a verdant hillside, in a remote part of Turkey attracts students from high school onwards to enjoy mathematics in beautiful environs, with no pressure of exams. The underlying principle is that it is more important to understand the problem than to just solve it. See [16] for more on the Nesin mathematical Village.

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8 An invitation to Ehrhart theory of lattice points in polytopes *by* J K Verma

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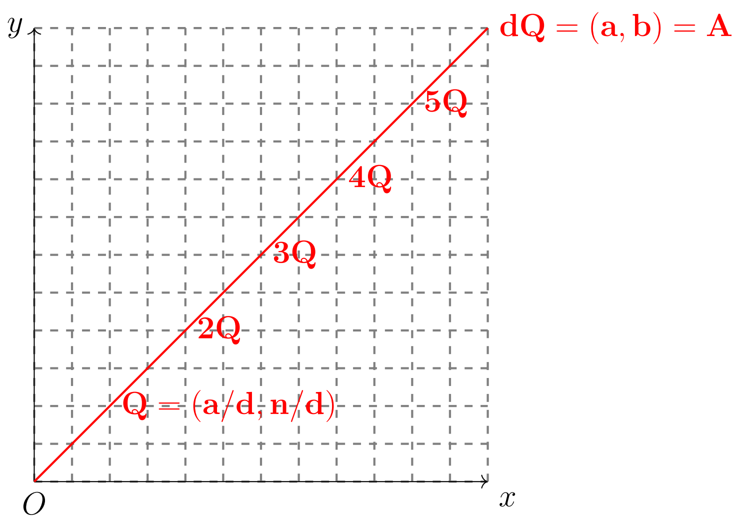
Abstract.[2] We shall introduce Eugène Ehrhart's Theory of lattice points in polytopes through several examples and show its beautiful connections with generating functions, Bernoulli polynomials, classical theorem of George Pick, counting magic squares and Ehrhart-Macdonald Reciprocity Theorem.

Introduction. A point whose coordinates are integers is called a lattice point. In these notes we discuss several examples of enumeration of lattice points which point to the general theorems discovered by E. Ehrhart and I. G. Macdonald. We will also explain how this theory throws light on the problem of counting magic squares. Let us begin with counting lattice points on a line segment.



[2]This is an expository article based on a lecture delivered to high school students and teachers who assembled for the a function for Mathematics Olympiad students at HBCSE in 2018. The material, taken from the references, is standard. No claim of originality is made.

Lemma 8.1. *Let $A = (a, b)$ be a lattice point and O denote the origin. Let $d = \gcd(a, b)$ and $Q = (a/d, b/d)$. Then the lattice points lying on the line segment OA are exactly the points*

$$Q, 2Q, \ldots, dQ = (a, b).$$

Proof. The equation of the line L joining O and A is $y = \frac{b}{a}x$. The point $Q = (\frac{a}{d}, \frac{b}{d})$ and its multiplies $Q, 2Q, \ldots, dQ = (a, b)$ lie on OA. If $(p, q) \in OA$ is a lattice point then $q = \frac{b}{a}p$. Thus $\frac{a}{d}q = \frac{b}{d}p$. But then $p = n\frac{a}{d}$ for some n and hence $q = \frac{b}{d}n$. Hence $(p, q) = n(a, b)$ Therefore the lattice points on OA are $O, Q, 2Q, 3Q, \ldots, dQ = (a, b)$. Similarly the number of lattice points on tOA is $td + 1$. Notice that it is a linear function of t and a line segment is a one-dimensional lattice polytope. □

Theorem 8.2 (Georg Pick, 1899). *Let P be a lattice polygon in the plane. Let $B =$ the number of lattice points on the boundary of P. Let $I =$ the number of lattice points in the interior of P. Then*

$$\textit{Area}\ (P) = (I - 1) + \frac{B}{2}.$$

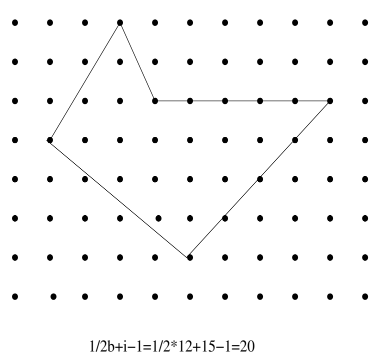
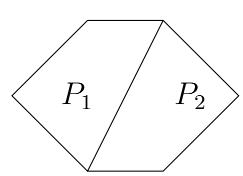


Figure 3: A lattice polygon



Let P be the lattice polygon. The function $area(P) = \frac{1}{2}B + I - 1$ is additive in nature. This means, if we divide P into two disjoint polygonal regions P_1 and P_2 so that $P = P_1 \cup P_2$ then

$$area(P) = area(P_1) + area(P_2) = \frac{1}{2}B_1 + I_1 - 1 + \frac{1}{2}B_2 + I_2 - 1$$

where $B_i =$ (resp. I_i) number of lattice points on the boundary (resp. interior) of P_i for $i = 1, 2$. Let L be the number of lattice points on the common boundary of P_1 and P_2. Then

$$\begin{aligned} I &= I_1 + I_2 + L - 2 \\ B &= B_1 + B_2 - 2L + 2 \\ I - 1 + \frac{1}{2}B &= I_1 + I_2 + L - 3 + \frac{1}{2}(B_1 + B_2 - 2L + 2) \\ &= (I_1 + \frac{1}{2}B_1 - 1) + (I_2 + \frac{1}{2}B_2 - 1) \end{aligned}$$

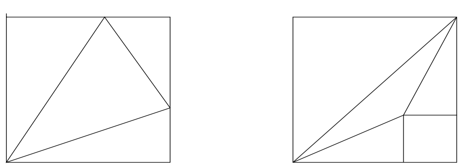


Figure 4: Embedding a triangle into a rectangle

Definition 8.3. *A Lattice triangle that has only vertices as its lattice points is called a* **fundamental triangle***.*

We can now cover the area by **fundamental lattice triangles.** Hence it is enough to derive the formula for these lattice triangles. We surround these triangles by integral rectangles. So it is enough to prove the theorem for rectangles and triangles whose sides are parallel to the axes and the hypotenuse has only two lattice points.

Theorem 8.4. *Let $\triangle ABC$ be a lattice triangle that has no interior lattice points and the vertices are the only lattice points on its boundary. Then*

$$area(\triangle ABC) = \frac{1}{2}.$$

Theorem 8.5. *Let P be a lattice polygon with area A and let B be the number of lattice points on its boundary. Let $L_P(t)$ denote the number of lattice points in tP. Then*
(1) $L_P(t) = At^2 + \frac{1}{2}Bt + 1$.
(2) *Let P° denote the interior of P. Then*

$$L_{P^\circ}(t) = (-1)^{\dim P} L_P(-t).$$

Proof. (1) Since $\text{area}(P) = I - 1 + \frac{1}{2}B$, The number of lattice points in P is

$$I + B = \text{area}(P) + 1 - \frac{1}{2}B + B = 1 + \text{area}(P) + \frac{1}{2}B.$$

Since $\text{area}(tP) = t^2\,\text{area}(P)$ we get the formula.
(2) Since $I = \text{area}(P) - \frac{1}{2}B + 1$, we get

$$L_{P^\circ}(t) = t^2\,\text{area}(P) - \frac{1}{2}tB + 1 = L_P(-t).$$

□

Theorem 8.6. *Let P be a d-dimensional polytope in $\mathbb{R}^d$. Then*

$$L_P(t) = \text{vol}(P)t^d + \textit{ lower degree terms in } t + \cdots.$$

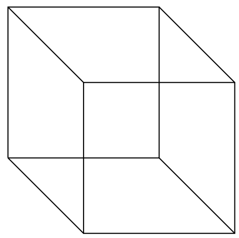
Proof. There is a one-to-one correspondence between $tP \cap \mathbb{Z}^n$ and $P \cap \frac{1}{t}\mathbb{Z}^n$ given by $u \longleftrightarrow \frac{1}{t}u$. If $u \in P \cap \frac{1}{t}\mathbb{Z}^n$, we draw hypercube with of side length $1/t$ with centre u. The number of such hypercubes is $L_P(t)$ and the volume of each such hypercube is t^{-d}. As $t \to \infty$, the total volume of these hypercubes approaches the volume of P. Hence $L_P(t)t^{-d} \to \text{vol}(P)$. Therefore

$$L_P(t) = \text{vol}(P)t^d + \dots.$$

□

Lattice point enumerator of simplex, cube and pyramid

Example 8.7 (The lattice point enumerator of the unit cube). *Consider the unit cube* C :

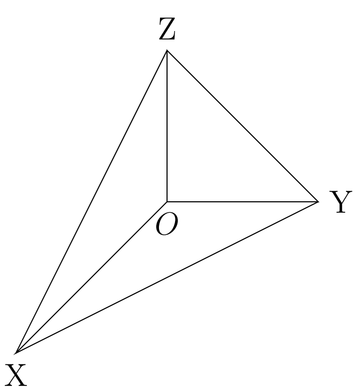


The lattice point enumerator of the unit cube is

$$L_C(t) = (t+1)^3 \text{ and } L_{C^\circ}(t) = (t-1)^3 = (-1)^3 L_C(-t).$$

Example 8.8 (The lattice point enumerator of the standard simplex). The standard simplex Δ of dimension d is the convex hull of the origin and the points $e_1, e_2, \ldots, e_d$ where e_j is the vector with 1 in the j^{th} position and 0 elsewhere. For $d = 3$, we have

$$\Delta = \{(x, y, z) \mid x + y + z \leq 1 \text{ and } x, y, z \geq 0\}.$$



The The dilated standard simplex in dimension d is the set

$$t\Delta = \{(x_1, x_2, \ldots, x_d) \mid x_1 + x_2 + \cdots + x_d \leq t \text{ and all } x_j \geq 0\}.$$

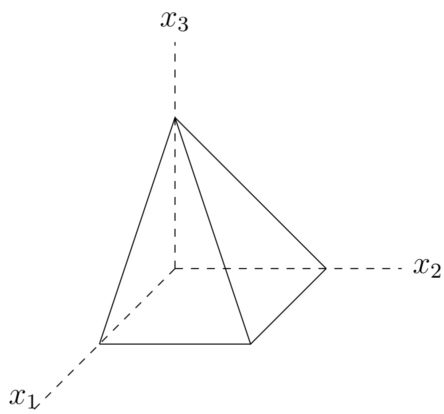
The lattice points in $t\Delta$ are integer solutions to the inequality $m_1 + m_2 + \cdots + m_d \leq t$. Therefore

$$L_\Delta(t) = \binom{t+d}{d}.$$

The lattice points in the interior of Δ are positive integers which are solutions to the inequality $m_1 + m_2 + \cdots + m_d < t$. This happens to be

$$L_{\Delta^\circ}(t) = \binom{t-1}{d} = (-1)^d \binom{-t+d}{d} = (-1)^d L_\Delta(t).$$

Example 8.9 (The Lattice point enumerator of the Pyramid). Take a $(d-1)$-dimensional unit cube embedded into $\mathbb{R}^d$ and adjoin the vertex at the point e_d.



The pyramid is also described as

$$\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid 0 \le x_1, x_2, \ldots, x_{d-1} \le 1 - x_d \le 1\}.$$

The lattice points in tP are

$$\{(m_1, m_2, \ldots, m_d) \in \mathbb{N}^d \mid m_k + m_d \le t \text{ for all } k = 1, 2, \ldots, (d-1)\}.$$

Each m_k for $k = 1, 2, \ldots, d-1$ has $t - m_d + 1$ independent choices. Hence

$$L_P(t) = \sum_{m_d=0}^{t} (t - m_d + 1)^{d-1} = \sum_{k=1}^{t+1} k^{d-1}.$$

To find the last sum, we recall the definition of the Bernoulli polynomials. First recall the exponential function

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \ldots.$$

The Bernoulli polynomials, named after Jacob Bernoulli (1654-1705) are defined through the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!}.$$

The first few Bernoulli polynomials are

$$\begin{aligned}
B_0(x) &= 1, \\
B_1(x) &= x - \frac{1}{2}, \\
B_2(x) &= x^2 - x + \frac{1}{6}, \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, \\
B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,
\end{aligned}$$

The **Bernoulli numbers** are $B_k = B_k(0)$ have the generating function

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}.$$

Theorem 8.10. *For $d \geq 1$ and $n \geq 2$, we have*

$$\sum_{k=0}^{n-1} k^{d-1} = \frac{1}{d} \left[B_d(n) - B_d(0) \right].$$

Proof. Calculate the generating function of the sequence $\frac{1}{d!}\left[B_d(n) - B_d(0)\right]$.

$$\begin{aligned}
\sum_{d\geq 0} \frac{B_d(n) - B_d}{d!} z^d &= z\frac{e^{nz}-1}{e^z-1} = z\sum_{k=0}^{n-1} e^{kz} \\
&= z\sum_{k=0}^{n-1}\sum_{j\geq 0}\frac{(kz)^j}{j!} \\
&= \sum_{j\geq 0}\left(\sum_{k=0}^{n-1} k^j\right)\frac{z^{j+1}}{j!} \\
&= \sum_{j\geq 1}\left(\sum_{k=0}^{n-1} k^{j-1}\right)\frac{z^j}{(j-1)!}
\end{aligned}$$

Compare the co-efficients of z^d on both the sides to get the formula. □

Using the above formula we can write the lattice point enumerator of the d-dimensional pyramid

$$L_P(t) = \frac{1}{d}\left(B_d(t+2) - B_d\right).$$

This is a polynomial of degree d in t whose leading coefficient is $1/d$ which is the volume of the pyramid.

Let us now find the function $L_{P^\circ}(t)$. The set of lattice points in the interior of P is

$$\{(m_1, m_2, \ldots, m_d) \in \mathbb{N}^d \mid 0 < m_k < t - m_d < t \text{ for all } k = 1, 2, \ldots, d-1\}.$$

Therefore

$$L_{P^\circ}(t) = \sum_{m_d=1}^{t-1}(t - m_d - 1)^{d-1} = \sum_{k=0}^{t-2} k^{d-1} = \frac{1}{d}(B_d(t-1) - B_d).$$

The Bernoulli polynomials satisfy the relation $B_d(1-x) = (-1)^d B_d(x)$. Moreover $B_d = 0$ for all odd $d \geq 3$. Hence we get

$$\begin{aligned}
L_P(-t) &= \frac{1}{d}(B_d(-t+2) - B_d) = \frac{1}{d}(B_d(1-(t-1)) - B_d) \\
&= (-1)^d\frac{1}{d}(B_d(t-1) - B_d) = (-1)^d L_{P^\circ}(t)
\end{aligned}$$

Magic squares and theorems of Eugène Ehrhart and I. G. Macdonald

The properties of $L_P(t)$ we observed for the cube, the standard simplex, polygons in the plane and pyramids are special cases of the theorems of Ehrhart and MacDonald. These are valid for polytopes.

Definition 8.11. *A subset S of $\mathbb{R}^n$ is called* convex *if for any two points $p, q \in S$, the line segment joining p and q is contained in S. The convex hull of a finite set of points in $\mathbb{R}^n$ is called a polytope. A lattice point is a point whose coordinates are integers. A polygon is called a lattice polygon if its vertices are lattice points. For a positive integer t and a polytope P, the set*

$$tP = \{tu \mid u \in P\}$$

is also a polytope.

Theorem 8.12 (E. Ehrhart). *If P is a d-dimensional integral polytope then $L_P(t)$ is a polynomial of degree d with rational coefficients.*

Theorem 8.13 (Ehrhart-Macdonald Reciprocity). *Suppose P is a rational convex polytope. Then*

$$L_P(-t) = (-1)^{\dim P} L_{P^\circ}(t).$$

Eugène Ehrhart (1906-2000) was a French mathematician who taught in high schools in France. He received his Ph. D. degree from the university of Strasbourg at the age of 60.

Counting magic squares

We shall now discuss the problem of counting magic squares. An $n \times n$ magic square is an $n \times n$ matrix whose entries are nonnegative integers and the sums of entries in every row and column is a given integer called the line sum. We introduce the function

$$H_n(t) = \text{ the number of } n \times n \text{ magic squares with line sum } t.$$

Observe that $H_n(0) = 1, H_n(1) = n!, H_2(t) = t + 1$

Theorem 8.14 (H. Anand, V. C. Dumir and H. Gupta, 1966).

$$\sum_{n=0}^{\infty} \frac{H_n(2)x^n}{(n!)^2} = \frac{e^{x/2}}{\sqrt{1-x}}.$$

P. A. MacMahon found a formula in 1960 for $H_3(t)$.

$$H_3(t) = \binom{t+4}{4} + \binom{t+3}{4} + \binom{t+2}{4}.$$

Guided by this evidence, Anand-Dumir-Gupta proposed the following conjecture.

Conjecture 8.15 (Anand-Dumir-Gupta, 1966). *$H_n(t)$ has the following properties:*
(1) *The function $H_n(t)$ is a polynomial in t of degree $(n-1)^2$.*
(2) *$H_n(j) = 0$ for $j = -1, -2, \ldots, -(n-1)$.*
(3) *$H_n(-n-t) = (-1)^{n-1}H_n(t)$.*

Proof. (1) If $M = (a_{ij})$ is a magic square with line sum t then $0 \leq a_{ij} \leq t$ and if a_{ij} is given for $i, j = 1, 2, \ldots, n-1$ then the remaining entries are uniquely determined. Hence

$$H_n(t) \leq (t+1)^{(n-1)^2}, \text{ so } \deg H_n(t) \leq (n-1)^2.$$

On the other hand if we choose any a_{ij} for $i, j = 1, 2, \ldots, n-1$ which satisfy

$$\frac{(n-2)t}{(n-1)^2} \leq a_{ij} \leq \frac{t}{(n-1)^2}$$

then we can determine the other entries of M. Hence

$$H_n(t) \geq \left(\frac{t}{n-1} - \frac{(n-2)t}{(n-1)^2}\right)^{(n-1)^2} = \left(\frac{t}{(n-1)^2}\right)^{(n-1)^2}.$$

Hence $\deg H_n(t) \geq (n-1)^2$. Therefore $\deg H_n(t) = (n-1)^2$.

(2) Consider the set of $n \times n$ doubly stochastic matrices:

$$B_n = \left\{ \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix} \mid x_{ij} \geq 0, \text{ and } \sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1 \right\}$$

B_n is called the Birkhoff polytope. The $n \times n$ permutation matrices are magic squares with line sum 1. Let these matrices be denoted by $P_1, P_2, \ldots, P_d$ where $d = n!$. Then for any $n_1, n_2, \ldots, n_d \in \mathbb{N}$,

$$n_1P_1 + n_2P_2 + \cdots + n_dP_d$$

is a magic square with line sum $n_1 + n_2 + \cdots + n_d$. G. D. Birkhoff and and von-Neumann showed that the permutation matrices are the vertices of the polytope B_n. Moreover its dimension is $(n-1)^2$. Hence by Ehrhart's Theorem, it follows that

$$H_n(t) = L_{B_n}(t) \text{ and } \deg H_n(t) = (n-1)^2.$$

The interior of B_n consists of $n\times n$ doubly stochastic matrices with positive entries. Let J_n denote the $n\times n$ matrix whose each entry is 1. Then we have a one-to-one correspondence between lattice points in tB_n and $(t+n)B_n^\circ$ given by $M \longleftrightarrow M + J_n$. By the Ehrhart-Macdonald Reciprocity Theorem we have

$$H_n(t-n) = L_{B_n^\circ}(t) = (-1)^{(n-1)^2} L_{B_n}(-t) = (-1)^{(n-1)^2} H_n(-t).$$

Since $L_{B_n^\circ}(t) = 0$ for all $t = 1, \ldots, (n-1)$, it follows that

$$H_n(t) = 0 \text{ for } t = -1, -2, \ldots, -(n-1).$$

This proves the second part of the ADG conjecture. □

Corollary 8.16 (P. A. MacMahon, 1960). *The number of* 3×3 *magic squares is*

$$H_3(t) = \binom{t+4}{4} + \binom{t+3}{4} + \binom{t+2}{4}.$$

Proof. We know that $H_3(t)$ is a polynomial of degree 4. It is enough to know its values at 4 different integers to determine its coefficients. Using the values

$$H_3(1) = 6, H_3(0) = H_3(-3) = 1, H_3(-1) = H_3(-2) = 0,$$

one can easily prove MacMahon's formula. □

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9 The Higher Arithmetic by H. Davenport - Book Review *by* Anupam Saikia

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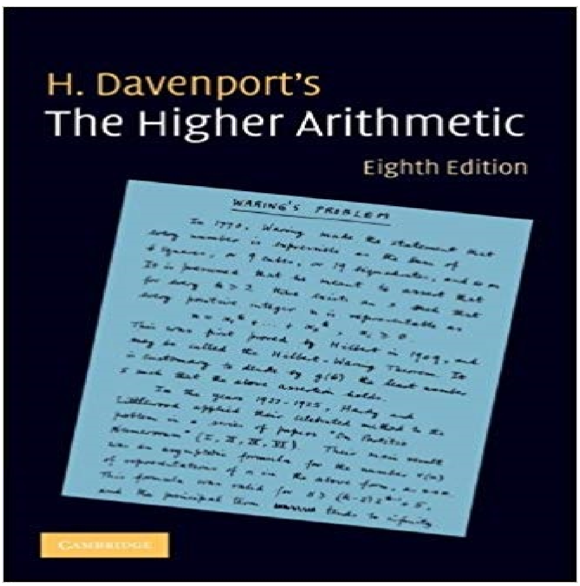


Harold Davenport (1907–1969) was an eminent British mathematician who made outstanding contribution to geometry of numbers, Diophantine approximation and the analytic theory of numbers. He wrote *The Higher Arithmetic* as an introduction to number theory for a general audience. The first edition of the book was published by Cambridge University Press in the year 1952. The book has undergone several editions and reprints afterward testifying to its enduring appeal. It introduces the readers to the theory of numbers in an engaging expository manner. It does not require its readers to have an extensive prior knowledge in mathematics. In fact it suffices to have a good high-school training in mathematics to follow this book. At the same time, the book throws light on topics of genuine mathematical significance in a truly enjoyable way. It is an immensely readable, stimulating and rewarding book for a wide variety of readers.

The eight edition of *The Higher Arithmetic* contains 239 pages which have been divided into eight chapters. The first chapter discusses elementary topics such as factorization of integers and Euclid's algorithm before alluding to some of the open questions concerning distribution and representation of primes. The second chapter deals with the notion of congruence and is of elementary nature too. The third chapter talks about primitive roots and quadratic residues providing a thorough treatment of quadratic reciprocity law. The fourth chapter provides a comprehensive introduction to the theory of continued fractions. The approximation properties of convergents have been highlighted too. Starting with the basics, this chapter gradually builds up the proof of Lagrange's theorem that an irrational number of the form $\sqrt{N}$ has a continued fraction which is periodic after a certain stage. The fifth chapter is an elegant discussion on representation of integers as sum of two, three and four squares. Lagrange's theorem that any positive integer can be represented as sum of four squares is beautifully explained here. The sixth chapter discusses quadratic forms, equivalent forms and representation of integers by them. It introduces the notion of class number as the cardinality $C(d)$ of equivalent classes of quadratic forms of a given discriminant d before touching on the unresolved conjecture of Gauss on existence of infinitely many positive integers d such that $C(d) = 1$. The seventh chapter deals with some of the very well-known Diophantine equations and also introduces the basic notion of elliptic curves. The final chapter, a later addition to the original book, discusses several factorization methods, primality testing, RSA cryptography etc. At the end, there is a list of well-chosen exercises followed by hints to their solutions.

The book is written very elegantly. It is not written in a rigid style of statement of results to be followed by proofs and applications. The exposition in the book is clear and precise. Without even being conscious of it, the readers are likely to get drawn from the elementary notions into deeper structures and questions. One can also gain a historical perspective about the development of the theory.

In the reviewer's opinion, any undergraduate who is interested in mathematics, and number theory in particular, will benefit immensely by going through *The Higher Arithmetic*. But many undergraduate students of mathematics in India are seemingly unaware of this book. One of the reasons may be that the book is not often mentioned in the list of reference books in the undergraduate curriculum of many universities in India. Hence the reviewer feels that that the book should be brought to the attention of undergraduates with a liking for number theory. Though it was not written as a textbook, it can be followed as one too. The book has stood the test of time. It has enthralled several generations of readers and will continue to do so. In the reviewer's opinion, this book is a must read for anyone interested in stepping into the beautiful world of number theory.

10 Quizzeria - Problems and Puzzles

Puzzles

In this part, we pose some puzzles whose answers are given on the last page.

• *Who is being referred to here and what is his age?*

> Here lies ....., the wonder behold.
> Through art algebraic, the stone tells how old:
> 'God gave him his boyhood one-sixth of his life,
> One twelfth more as youth while whiskers grew rife;
> And then yet one-seventh ere marriage begun;
> In five years there came a bouncing new son.
> Alas, the dear child of master and sage
> after attaining half the measure of his father's life chill fate took him.
> After consoling his fate by the science of numbers for four years,
> he ended his life.'

• What do the following mean?

More toys hate graph (Hint: Already in Sulvasutras)
Is ISI price done? (Hint: What goes around comes around)
Steel right as rain (Hint: Never deviate from your path)
Stir in nascent pedal (Hint: Like 'e' but not like $\sqrt{2}$)
Tonic is army gain (Hint: Not real but false).

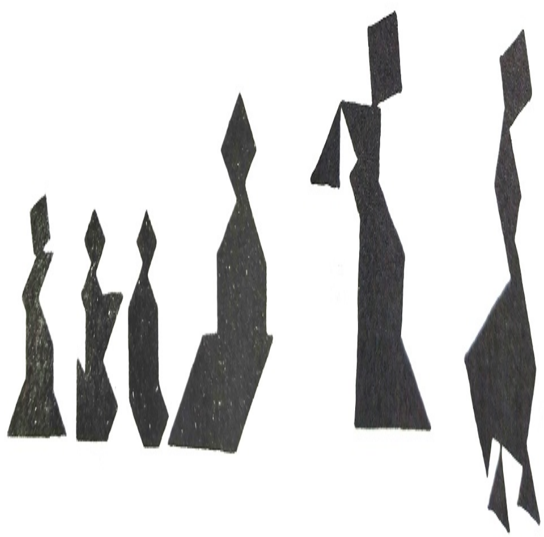
• While on convalescent leave from service in World War I, this mathematician killed his brother, his aunt and his uncle. He told the medical head of the asylum where he was confined that the murders were a eugenic act, in order to eliminate branches of his family affected by mental illness. This person also had the habit of dating his letters with 1st April, regardless of when they were written. Who was this?

• This woman mathematician was born on 1st April and, using the pseudonym M Le Blanc, managed to obtain lecture notes for courses from the newly organized Ecole Polytechnique in Paris. Who was this mathematician?

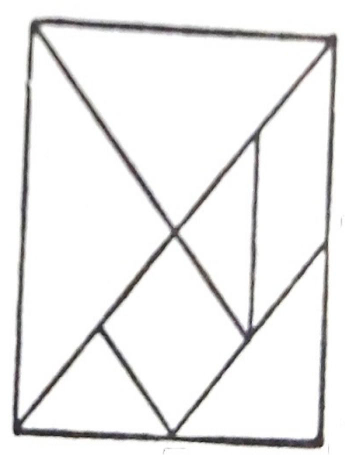
• Who said the following and who is being referred to?
"He is a man of good birth and excellent education, endowed by nature with a phenomenal mathematical faculty. At the age of twenty-one, he wrote a treatise upon the binomial theorem, which has had a European vogue. On the strength of it he won the mathematical chair at one of our smaller universities, and had, to all appearances, a most brilliant career before him."

• Hermite was the major mathematical authority in France in the 1880's. He crusaded for the career of his three mathematical stars: Paul Appell (his nephew), Emile Picard (his son-in-law) and a third person. Hermite wrote to Mittag-Leffler that he considered this third person to be the most brilliant, though he did not belong to his family, to the great displeasure of Madame Hermite. Who is this third person?

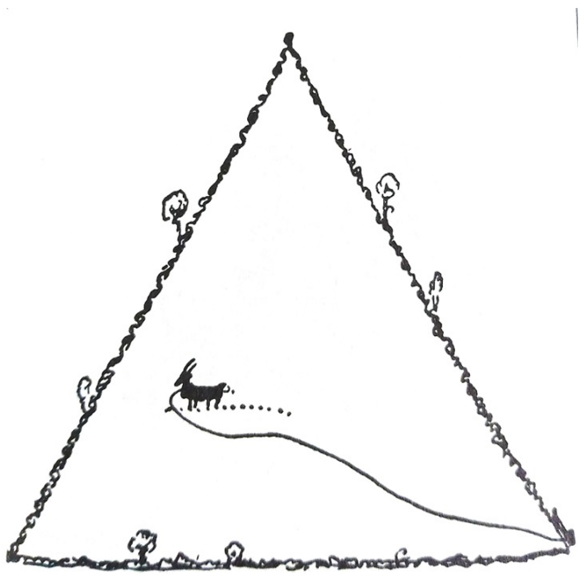
• Construct the following shapes (solutions in next issue)



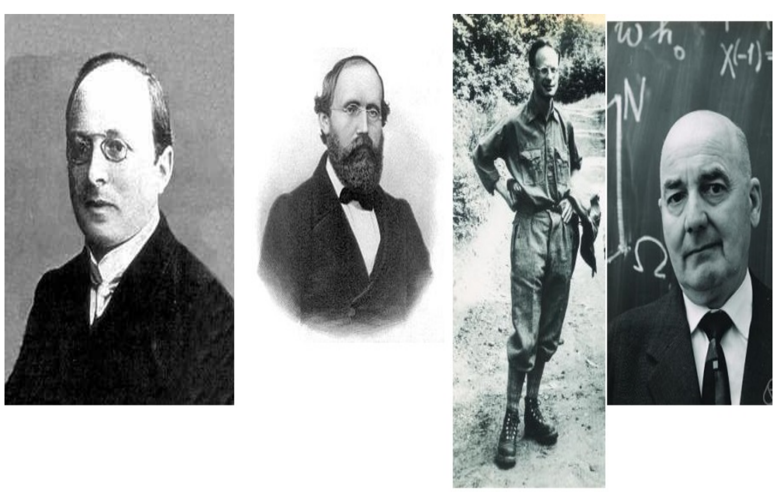
using the 7 pieces below:



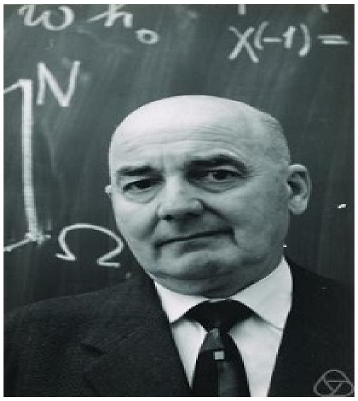
• This puzzle is due to Henrey Dudeney. A goat is tethered to a vertex of an equilateral triangle. It grazes as much area as covered by the sweep of the rope and this happens to be 6π square metres. What is the length of the rope?

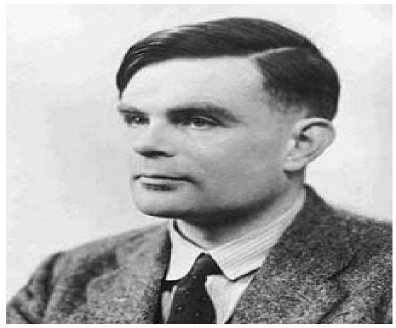


• What is common to the following four mathematicians?



• Arrange the following mathematicians in some natural pattern.





• The mathematician on the left is Herglotz. The one on the right lost his nose during World War I. Who is he?

Solutions to the problems below are invited. The first six are easier and the rest are in increasing order of difficulty. The solutions will appear in the next issue.

Problems

Q 1. Solve ACID + BASE = SALT + H2O where each letter stands for a distinct digit.

Q 2. An $l \times b$ rectangular painting needs to be covered by a square frame. Find the side length of the smallest square frame required.

Q 3. Find the smallest positive integer N so that the sum $\sum_{n=1}^{N} \frac{1}{n!+(n+1)!} > 0.49995$.

Q 4. In the multiplication table below, each digit from 0 to 9 appears exactly twice. Determine all such tables.

		*	*	*
	×	*	*	*
		*	*	*
	*	*	*	
*	*	*		
*	*	*	*	*

Q 5. Determine the sum of the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \cdots$ mentioned on the cover page.

Q 6.
(a) Prove the identity $\sum_{n \geq 1} \frac{1}{n^n} = \int_0^1 \frac{dx}{x^x}$ mentioned on the cover page.
(b) Prove the identity $\sum_{n \geq 1} \frac{(-1)^{n-1}}{n^n} = \int_0^1 x^x dx$ mentioned on the cover page.

Q 7. Let f be a function from $\mathbb{R}$ to itself. Suppose that for each $a \in \mathbb{R}$, the number of elements in $\{t \in \mathbb{R} : f(t) = a\}$ is 0 or 2. Prove that f has infinitely many points of discontinuity.

Q 8. A deck of 52 cards is given. There are four suites each having cards numbered $1, 2, \cdots, 13$. The audience chooses any five cards with distinct numbers written on them. The assistant of the magician comes by, looks at the five cards and turns exactly one of them face down and arranges all five cards in some order. Then the magician enters and with an agreement made beforehand with the assistant, she has to determine the face down card (both suite and number). Explain how the trick can be completed.

Q 9. Find all polynomials p which satisfy the property that a value $p(a)$ is rational if and only if a is rational.

Q 10. Let G be a finite group of order n. For any subset S of G, put

$$S^k = \{s_1 \cdots s_k : s_i \in S\}$$

for each $k \geq 1$. Prove that S^n is always a subgroup.

Q 11. Let us mark off points on the unit circle, dividing the circumference into n equal parts where $n > 2$. Fix one of these points and, moving clockwise along the circumference, join this point to the k-th point for each k coprime to n. What is the products of the lengths of these chords?

Q 12. Wieferich observed in 1909 that if p is a prime for which Fermat's equation $x^p + y^p = z^p$ has a solution in positive integers which are not multiples of p, then $2^{p-1} \equiv 1 \bmod p^2$. Such primes are called Wieferich primes. Prove that a prime which is Mersenne or Fermat cannot be a Wieferich prime. Here Mersenne primes are those of the form $2^n - 1$ and Fermat primes are those of the form $2^m + 1$.

Q 13. Prove that there is no function $f : \mathbf{R} \to \mathbf{R}$ which is differentiable at every irrational number and discontinuous at every rational number.

Q 14. Find the significance of the number 4959866698915122609810424451291 8.

Q 15. Determine whether the series $\sum_n \frac{1}{n^3 \sin^2(n)}$ converges or not.

11 Math Humour, Cross-Swords

• Math Humour

When Galois rebelled against the establishment, he was ineffective in solving the problem; why?
Because the problem was not solvable by radicals.

Why is the usual formula for the area of a circle wrong?
Who says pie are square; they are round.

Ancient Roman in a cloth store: "how come XL is larger than L?"

There are three types of people in the world: those who can count and those who can't.

(*Due to Michael Filaseta*) *82.1032 per cent of statistics are made up on the spot.*

(*From Hendrik Lenstra's classes*):
"Who doesn't know what a local ring is? Don't be shy... [Student raises his hand] Learn it!"
"So h(x) must be the inverse of zero. This is a very big problem for x, which decides to solve this problem by ceasing to exist."
"That would be an interesting problem to think about. Of course, the first thing to do is to turn the library upside down and see if something falls out."
"Adèles. You put the accent there if you want people to know you speak French."
"The problem with wrong proofs to correct statements is that it is hard to give a counterexample."
"The art of doing mathematics is forgetting about the superfluous information."
"Recreational Number Theory is that branch of Number Theory which is too difficult for serious study."
"I'll show you how to discover all of this by yourself, assuming that you are Fermat."

• **Cross-Sword**

1			2	♠	3	4		5	6
	♠	♠		♠	♠		♠	7	
	♠	♠	8	9	10		♠	♠	
11		12	♠	13		♠	14	15	
	♠	16	17	♠	♠	♠	18		
♠	♠	19		♠	♠	♠		♠	
20			♠	21	22	♠	23		
	♠	♠	24			25	♠	♠	
26	27	♠		♠	♠		♠	28	
29					♠	30			

Across

1. Nasty but average (4).
3. Euler's constantly talking Greek (5).
7. Cures a Ph.D? (2).
8. It's complex without an iota of doubt (4).
11. The least upper bound may not be attained but briefly (3).
13. If not General Motors, it is $\sqrt{ab}$ (2).
14. And so on? (3).
16. $a, a+b, a+2b, \cdots$ (2).
18. Cartoon character makes a point (3).
19. A couple of egotists could make a cricket team (2).
20. Follows from the theorem - at least in the beginning (3).
21. Romans counting in Greek? (2).
23. Any trigonometric function under the sun gets this (3).
24. Keep thinking twice to be level (4).
26. Greek novel? (2).
28. 49 is almost ill (2).
30. A function that is little more than a sin (4).

Down

1. The pet of computer enthusiasts (5).
2. This follows neither (3).

4. Or nothing? (3).
5. Could be 7 across (2).
6. Disturbed clear agent has four sides (1,9).
9. For instance (2).
10. This is at least as large as 13 across (2).
12. One of these could be two of these in Hindi (4).
14. A change in diet before publication (4).
15. Following 28 down could give us a product (2).
17. Pig without tail goes around in circles (2).
20. A function drawn without lifting the pen briefly (4).
21. Sum of digits of all its divisors is equal to it. (2).
22. That is to say (2).
24. Shortened member in a group (3).
25. Count them as negative answers? (3).
27. Unitary operators on a Hilbert space simply call for an exclamation (2).
28. See 15 down (2).

12 Are these beautiful? *by* Kanakku Puly

Bures Sur Yvette

G.H.Hardy says, "The mathematician's patterns, like the painter's or the poet's, must be beautiful.... Beauty is the first test: there is no permanent place in the world for ugly mathematics." Von Neumann, Poincare as well as Weyl have expressed similar sentiments. Weyl goes a step further when he says that although in mathematics he looks for truth and beauty, when he has to make a choice, he may choose the latter! We discuss a few of the most beautiful results from mathematics. Also, in some cases, rather than the result itself, it is a particular proof which one thinks of as beautiful. One further point to note is that not all these facts have the same depth. Although the perception of beauty in a mathematical result is rather subjective, irrespective of whether a result is easy or difficult to prove, it appears that what is considered by most people to be beautiful mathematics has an element of surprise.

1. *The number of primes is infinite.*
This fact itself is not as captivating as its famous immortal proof due to Euclid is. This single proof embodies in it the subtlety of the fundamental theorem of arithmetic. Although understandable to the proverbial layman, I have nevertheless not met anyone who has independently discovered this proof. Notice that $2.3.5.7.11.13 = 59.509$. Indeed, interestingly, in spite of enormous advances in prime number theory, it is still unknown whether the sequence $2.3.5 \cdots p_n + 1$ (where p_n is the n-th prime) contains infinitely many primes.

2. $\sqrt{2}$ *is irrational.*
This is perhaps the first time that a student encounters a proof (using the fundamental theorem of arithmetic) by contradiction. In the author's personal experience, students find the following proof using the method of descent much more surprising.
Suppose $\sqrt{2}$ is rational, say a/b with $a, b > 0$. Suppose b is the least possible natural number for which $b\sqrt{2}$ is an integer. But, then $b' = b(\sqrt{2} - 1)$ is a

natural number such that $b'\sqrt{2} = 2b - b\sqrt{2} \in \mathbb{Z}$. Clearly, $b' < b$ which contradicts the choice of b. By descent, this proves that $\sqrt{2}$ cannot be rational. Note that what we have proved is that

$$\sqrt{2} = \frac{a}{b} = \frac{2b-a}{a-b}.$$

The above proof by descent has the advantage that it easily generalises to prove that *a rational number which is a root of a monic integral polynomial must be an integer.* The above proof of irrationality of $\sqrt{2}$ can also be depicted by means of a diagram (try drawing one!).

3. (Euler :) $p_{odd}(n) = p_{distinct}(n)$ *for all* n.
In other words, the number of partitions of n into odd numbers is also the number of partitions of n into distinct numbers. This must rank among the top five favourites of all time in many lists!
This is an instance of a fact which is absolutely captivating on the face of it whereas the proof has no great surprises. For $|x| < 1$, note that

$$\prod_{n\geq 1}(1-x^n)^{-1} = \prod_{n\geq 1}(1+x^n+x^{2n}+\cdots).$$

The infinite product can also be shown to be convergent when $|x| < 1$. A typical term of the expanded product is of the form x^n with $n = r_1n_1 + r_2n_2 + \cdots + r_kn_k$ for some $n_1 < n_2 < \cdots < n_k$ and $r_1, \cdots, r_k \in \mathbb{N}$. Thus, the coefficient of x^n is the number $p(n)$ of partitions of n. Since a power series determines its coefficients, one may compare coefficients of like powers of x and obtain

$$\prod_{n\geq 1}(1-x^n)^{-1} = \sum_{n\geq 0} p(n)x^n.$$

Now, similarly

$$\prod_{n\geq 1}(1-x^{2n-1})^{-1} = \sum_{n\geq 0} p_{odd}(n)x^n.$$

Consider the infinite product $\prod_{n\geq 1}(1+x^n)$. Once again, it is evident from expanding this product that a power x^n occurs as many times as n can be written as a sum $n_1 + n_2 + \cdots + n_k$ of distinct natural numbers for some k. In other words,

$$\prod_{n\geq 1}(1+x^n) = p_{distinct}(n)x^n.$$

Now,

$$\prod_{n\geq 1}(1+x^n) = \frac{\prod_{n\geq 1}(1-x^{2n})}{\prod_{n\geq 1}(1-x^n)} = \frac{1}{\prod_{n\geq 1}(1-x^{2n-1})}.$$

We mention another identity; this can be proved in a geometric manner.
The number of partitions of n into at most m parts equals the number of partitions of n into parts in which each part is at most m.
This can be proved by actually producing a bijection between the two sets that one is counting. For a partition $n_1 + n_2 + \cdots + n_r = n$ where $n_1 \geq n_2 \geq \cdots \geq n_r$, draw an array consisting of dots with n_1 dots in the first row, n_2 dots in the second row (centred to the left) etc. as in figure 1. Associate to this array, the 'conjugate array' obtained by counting columnwise. For instance, $8 = 4 + 2 + 2$ gives the array in figure 1 and its conjugate array given in figure 2 corresponds to the partition $8 = 3 + 3 + 1 + 1$.
This operation of conjugation produces the bijection we are looking for.
The theory of partitions is a subject to which Ramanujan made many important contributions. Apart from their beauty and elegance, partition identities are intimately related to many subjects like statistical mechanics, representation theory, modular forms etc.

4. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
Whoever encounters this for the first time is invariably left with a sense of incredulity - how can a sum involving squares of natural numbers produce a number like π ?! Test this out by telling a friend who studies, let us say, chemistry or biology. However, not only is this true, it was already 'proved' by Euler. His 'proof' of this, in terms of today's rigour, is not satisfactory but can be made completely rigorous. In fact, the sum $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ for any even powers $2k$, is a rational multiple of π^{2k} where this rational number involves the so-called Bernoulli numbers. One can prove the above result in various ways but let us indicate what Euler himself did and take the possibility of its rigorisation for granted.
Think of the right hand side of $\frac{Sin(x)}{x} = 1 - x^2/3! + x^4/5! - \cdots\cdots$ as though it were a polynomial. Its zeroes are $\pm n\pi$ for natural numbers n. Now, if $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial whose roots are $\alpha_i \neq 0,\ i = 1, 2, \cdots, n$, then the polynomial $x^n f(1/x) = a_n + a_{n-1}x + \cdots + a_0$ has its

roots as $1/\alpha_i$. Therefore, the sum

$$\sum_{i=1}^{n} \frac{1}{\alpha_i^2} = (\sum_{i=1}^{n} \frac{1}{\alpha_i})^2 - 2\sum_{i<j} \frac{1}{\alpha_i \alpha_j} = (-a_1/a_0)^2 - 2a_2/a_0.$$

Use this idea for the above expression for $Sin(x)/x$, noting that $a_0 = 1, a_1 = 0, a_2 = -1/3!$. We have

$$\sum_{n \neq 0} \frac{1}{n^2\pi^2} = 2\sum_{n \geq 1} \frac{1}{n^2\pi^2} = \frac{2}{3!}.$$

Below, we have another such sum with π occurring:

$$\frac{1}{2\times 3\times 4} - \frac{1}{4\times 5\times 6} + \frac{1}{6\times 7\times 8} - \cdots\cdots = \frac{\pi - 3}{4}.$$

Although it may not be as surprising now as before, it is not immediately clear how such a thing could be proved. One can prove this by expanding as partial fractions as follows.

$$\sum_{k\geq 2} \frac{(-1)^k}{(2k-2)(2k-1)2k} = \sum_{k\geq 2}\{\frac{(-1)^k}{2(2k-2)} + \frac{(-1)^{k-1}}{2k-1} + \frac{(-1)^k}{4k}\}$$

$$= \frac{1}{4} + \sum_{k\geq 2} \frac{(-1)^{k-1}}{2k-1} = \frac{1}{4} + tan^{-1}(1) - 1 = \frac{\pi - 3}{4}.$$

5. $e^{i\pi} = -1$.

This beautiful formula figures at the top in most people's lists. In modern mathematics, it is the number e (or rather, the function e^x) which is central. One encounters π very early in life as a result of which often a beginner feels closer to π than to e. A famous theorem of Gelfond & Schneider asserts that α^β is not algebraic when $\alpha \neq 0, 1$ is algebraic and β is algebraic and irrational. Here, by a complex number being algebraic, one means that it is the root of some nonzero polynomial equation with rational number coefficients. The theorem of Gelfond & Schneider shows on using the above formula (a little cleverly) that e^π is not algebraic - one of Hilbert's famous problems. It is interesting to note that Gelfond & Schneider's theorem solved Hilbert's problem during his lifetime and that Hilbert had predicted initially that it would take many generations before it would be proved and that Fermat's last theorem and the Riemann hypothesis would be solved before that!

6. *There is no equilateral triangle with all vertices to be lattice points.*
Since the line joining two lattice points has rational slope, it follows that for any triangle with vertices as lattice points, tan θ for any of the three angles is rational. For an equilateral triangle tan $60 = \sqrt{3}$ which is irrational.

7. *At any party, there are at least two different people with the same number of friends.*
This is a very attractive application of the pigeon-hole principle. Here is a way to prove it. Write an $n \times n$ array with rows and columns indexed by the n people $P_1, \cdots, P_n$ in the party. Put the (i, j)-th entry to be 1 or 0 according as whether P_i and P_j are friends or not. Let us assume that each person is friendly with herself (!) so that all the diagonal entries are 1's. Then, the number of friends of P_i is the row sum r_i of the i-th row. Note that $1 \leq r_i \leq n$ for all i. If $r_1, \cdots, r_n$ were all distinct, then $r_1, \cdots, r_n$ would just be the numbers $1, \cdots, n$ in some order. But then if $r_i = 1$ and $r_j = n$, this means that P_i has no friends other than herself while P_j is friendly to everybody else. This apparent contradiction proves that $r_1, \cdots, r_n$ cannot all be distinct.

8. *If each point of the plane is coloured with one of three colours - red, yellow and blue - then there must exist two different points x, y at unit distance which have the same colour.*
The trick is to tile the whole plane by equilateral triangles of unit sides. In any such equilateral triangle, the vertices have to be coloured differently for, otherwise, we are through. But then, we have that any sequence of 3 consecutive edges gives rise to the extremes having the same colour. Thus, if we have A,B,C with AB and AC having length 3 and BC having length 1, then A and B have the same colour and so do A and C. Thus, B and C have the same colour!

9. The Cayley-Hamilton theorem : *Every square matrix A is a root of its 'characteristic' polynomial $p(t) = det(A - tI)$.*
This lovely and surprising result is often thought by students to be a tautology (what is wrong with the proof $p(A) = det(A - A) = 0$?!) A nice proof is the "engineers' style" method of "assuming what is to be proved and work-

ing backwards" to obtain a proof. In this instance, it can be carried out as follows. Write

$$p(t) = c_0 + c_1 t + \cdots + c_n t^n = det(A - tI).$$

Now, for each $t \in \mathbb{C}$, the adjoint of the matrix $A - tI$ can be written as

$$Adj(A - tI) = A_0 + A_1 t + \cdots + A_{n-1} t^{n-1}$$

where A_i are certain matrices which are independent of t. Now, the equality

$$(A - tI)Adj(A - tI) = p(t)I$$

for each $t \in \mathbb{C}$ implies
$AA_0 = c_0 I, AA_1 - A_0 = c_1 I, \cdots, AA_{n-1} - A_{n-2} = c_{n-1} I, -A_{n-1} = c_n I$.
Thus, 'working backwards', one may determine A_i's as polynomial expressions in A and thus the A_i's commute with A.
Thus, $p(A) = c_0 I + c_1 A + \cdots + c_n A^n = AA_0 + (AA_1 - A_0)A + (AA_2 - A_1)A^2 + \cdots + (AA_{n-1} - A_{n-2})A^{n-1} - A_{n-1}A^n = 0$, the zero matrix.
The reader is invited to check that this proof yields the more general result that if $AB = BA$, then $p(B) = (B - A)C$ for some matrix C.

13 Some elementary problems posed by Ramanujan *by* B Sury

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Here are some elementary problems from among the 58 problems posed by Ramanujan in the Journal of the Indian Mathematical Society between the years 1911 and 1919. I have chosen only those few which require very little background knowledge in order to understand the statements of the results. We mention hints for solving some of these problems and leave it to the reader to try her acumen to complete the solutions.

Q 1. *Solve the equation $x^y = y^x$ in positive rational numbers.*
The solution by J.C.Swaminarayan and R.Vythynathaswamy is very simple and goes as follows. Put $x/y = t$. Then $y^t = x = yt$ so that $y^{t-1} = t$. From this, it is not difficult to check that $t - 1 = \frac{1}{n}$ for some n (check !) Thus, $y = (1 + 1/n)^n$ and $x = (1 + 1/n)^{n+1}$ for any natural number n, are all possible positive rational solutions. Note that $\{x, y\} = \{2, 4\}$ are the only solutions in natural numbers. Actually, with some background in calculus one can solve the above problem by looking at the growth of the function $\frac{x}{log\ x}$.

Q 2. *Find the values of*

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}},$$

$$\sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + \cdots}}}.$$

Of course, it is not a trivial matter to analyse and justify that such an infinite sequence of 'nested radicals' gives a meaningful number. However, it is easy to find the values if one accepts that the expressions are meaningful and goes ahead. It is a pleasant exercise to prove that the values are 3 and 4 respectively! In fact, prove that

$$\sqrt{x+2\sqrt{x+1+3\sqrt{x+2+\cdots}}} = 1+\sqrt{x+3}.$$

Ramanujan also proved:
If m, n are arbitrary, then

$$\sqrt{m\sqrt[3]{4m-8n}+n\sqrt[3]{4m+n}} =$$

$$\pm\frac{1}{3}(\sqrt[3]{(4m+n)^2}+\sqrt[3]{4m-8n)(4m+n)}-\sqrt[3]{2(m-2n)^2}).$$

Actually, this is easy to verify simply by squaring both sides ! However, that does not indicate how this formula was arrived at or whether there are more general formulae. In fact, it turns out that Ramanujan was absolutely on the dot here; the following result shows Ramanujan's result cannot be bettered: *Let $\alpha, \beta \in \mathbb{Q}^*$ such that α/β is not a perfect cube in $\mathbb{Q}$. Then, $\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$ can be denested if and only if there are integers m, n such that $\frac{\alpha}{\beta} = \frac{(4m-8n)m^3}{(4m+n)n^3}$.*
The proof requires Kummer theory.

Q 3. *Show that*

$$\sqrt[3]{cos2\pi/7}+\sqrt[3]{cos4\pi/7}+\sqrt[3]{cos8\pi/7} = \sqrt[3]{(5-3\sqrt{7})/2}.$$

Anyone who has worked with number-theoretic and algebraic identities knows that, once written down, many identities are rather easy to verify but one often misses the creative insight which led to the discovery of the identity in the first place. For example, the above identity can be verified simply by taking 3rd powers and applying the multinomial theorem.

Q 4. *Show that*

$$(3\{(a^3+b^3)^{1/3}-a\}\{(a^3+b^3)^{1/3}-b\})^{1/3} = (a+b)^{2/3}-(a^2-ab+b^2)^{1/3}.$$

This problem was solved very elegantly by the fivesome - K.K.Ranganatha Aiyar, R.D.Karve, G.A.Kamtekar, L.N.Datta and L.N.Subramanyam. Here is the gist of their argument.
Consider the identity

$$(a+b-c)^3 = (a+b)^3-c^3-3c(a+b)^2+3c^2(a+b).$$

Taking $c^3 = a^3 + b^3$, the right hand side simply becomes $3(a+b)(c-a)(c-b)$. The asserted identity follows on dividing by $a + b$ and taking cube roots.

Q 5. *If*

$$sin(x + y) = 2sin((x - y)/2)\ ,\ sin(y + z) = 2sin((y - z)/2),$$

prove that

$$(sin(x)cos(z)/2)^{1/4} + (cos(x)sin(z)/2)^{1/4} = (sin(2y))^{1/12}.$$

It is astonishing to note that it took more than 10 years before the first solution was submitted. Even the latest solutions available are rather lengthy, and it is a challenge to find a shorter solution.

Q 6. $2^n - 7$ *is a perfect square for the values* $3, 4, 5, 7, 15$ *of* n. *Find other possible values.*
This is now known as the Ramanujan-Nagell equation. Interestingly, W.Ljunggren posed this same problem in 1943 unaware that Ramanujan had already done so; Nagell solved it in 1946. It turns out that the above values of n are the only ones for which $2^n - 7$ is a square. Now there are many proofs known of this fact but all of them involve non-elementary results in mathematics.

14 Answers to Puzzles

• The poem is about Diophantus's life. This puzzle implies that Diophantus's age $x = 84$ is a solution of the equation

$$x = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4.$$

• These are anagrams of:
Pythagoras theorem, Sine is periodic, Lines are straight, Pi is transcendental, C is not imaginary.

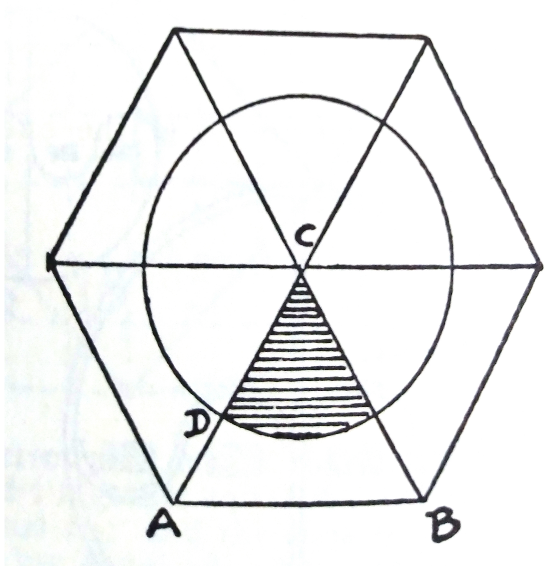
• Andre Bloch, who contributed to complex analysis from prison.

• Sophie Germain, who revealed her identity to Gauss later.

• Sherlock Holmes in 'The Final Problem' referring to his archrival Professor Moriarty.

• Henri Poincaré.

• The solution to the goat problem is obtained by viewing the equilateral triangle as one of the six that form the hexagon in the figure below. Then, the area covered is one-sixth of the area of the circle as in this figure. Thus, the radius is $\sqrt{6 \times 6\pi/\pi} = 6$ metres.



• Paul Epstein, Bernhard Riemann, Andre Weil and Helmut Hasse have some zeta functions named after them.

• Hermann Schubert, Alan Turing, Paul Epstein, and Helmut Hasse were born on 22/5, 23/6, 24/7, 25/8 respectively.

• With Gustav Herglotz is Gaston Julia who lost his nose during world war I.

Photos courtesy:

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$http://www-gap.dcs.st-and.ac.uk/\ history/Biographies/Schubert.html$
$http://www.turingarchive.org/viewer/?id = 521\&title = 4$
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Answer to Cross-Sword

M	E	A	N	♠	G	A	M	M	A
O	♠	♠	O	♠	♠	L	♠	D	R
U	♠	♠	R	E	A	L	♠	♠	E
S	U	P	♠	G	M	♠	E	T	C
E	♠	A	P	♠	♠	♠	D	O	T
♠	♠	I	I	♠	♠	♠	I	♠	A
C	O	R	♠	X	I	♠	T	A	N
O	♠	♠	E	V	E	N	♠	♠	G
N	U	♠	L	♠	♠	O	♠	I	L
T	H	E	T	A	♠	S	I	N	E

15 Objectives of Blackboard

The aim of Blackboard, the Bulletin of MTA (I) is to promote interest in mathematics at various levels and to facilitate teachers in providing a well-rounded mathematical education to their students, in curricular as well as extra-curricular aspects.. The Bulletin will also serve as an interface between MTA (I) and the broad mathematical community.

As a community of educators, we are also interested in learning from experiences of teachers. To be able to be effective in our practices of doing mathematics, we need to understand how students think and engage in these practices. We welcome write-ups from teachers that provide this window into student thinking from their actual teaching experiences. These glimpses will form a foundation of what we can offer as a useful resource for issues of teaching and learning of mathematics.

While there are other publications with similar objectives, Blackboard is envisaged to be different in terms of coverage of material and diverse target readership - high school and college teachers to young researchers. We plan to publish regularly on the following topics and themes:

- A variety of teaching initiatives.
- Articles connecting different stages of mathematics education and mathematical research.
- Expository articles on recent developments in mathematics.
- Classroom Practices.
- Issues of mathematics teaching and learning from actual teaching experiences.
- Work of Indian mathematicians.
- History of Mathematics.
- Problem Corner with puzzles, crosswords, cartoons, etc.
- Book Reviews.
- Information about useful online resources.
- Announcements of workshops, positions and other news. items.

Initially, Blackboard will be brought out as an e-publication.