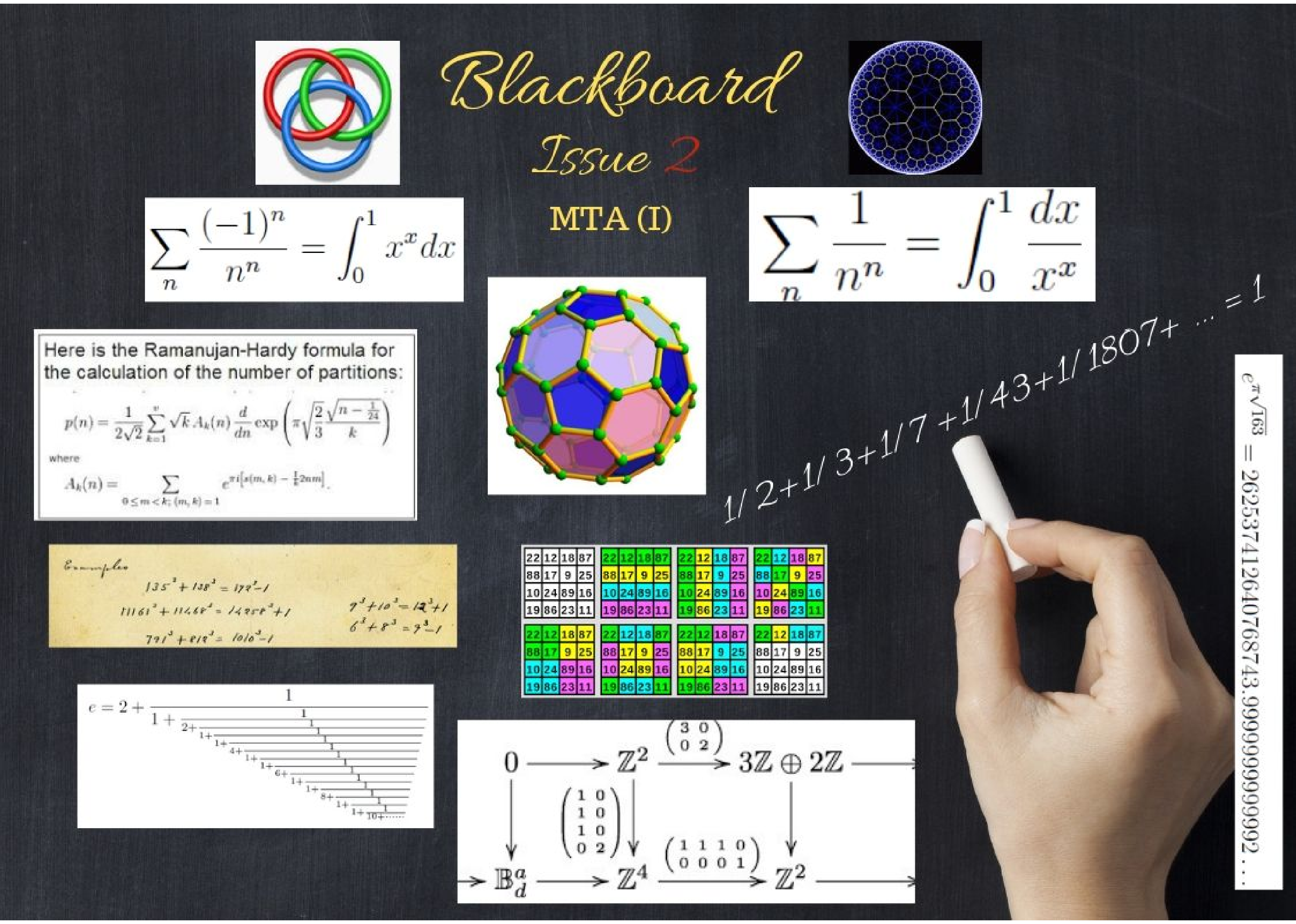
Blackboard
Issue 2
MTA (I)
$\sum_n \frac{(-1)^n}{n^n} = \int_0^1 x^x dx$
$\sum_n \frac{1}{n^n} = \int_0^1 \frac{dx}{x^x}$
Here is the Ramanujan-Hardy formula for the calculation of the number of partitions:
where
1/ 2+1/ 3+1/ 7 +1/ 43+1/ 1807+ ... = 1
$e^{\pi\sqrt{163}} = 262537412640768743.99999999999992\ldots$
Examples
$135^3 + 138^3 = 172^3 - 1$
$11161^3 + 11468^3 = 14258^3 + 1$
$791^3 + 812^3 = 1010^3 - 1$
$9^3 + 10^3 = 12^3 + 1$
$6^3 + 8^3 = 9^3 - 1$

Contents

1 Editorial

This is the second issue of Blackboard, the bulletin of the MTA(I). Its publication was delayed for a variety of reasons. For one thing, the editorial board was keen to hear feedback from members regarding how they liked the first issue and what further suggestions they had. We repeat that the Blackboard aims to be inclusive of mathematics teachers from all over India. Setting up the portals for members to log in and register their comments took much more time than the editorial board envisaged. Despite this long delay, there has not been much feedback excepting a handful which appreciated our efforts. However, we would really like suggestions from teachers as to what they would like to see published in the Bulletin.

As the inaugural conference of the MTA was a great success, attracting as it did, several participating teachers from schools as well as colleges from all over the country, it was decided to publish some of the talks given by them that were deemed suitable to appear in the Bulletin. In this issue, we carry three which, we hope, will entice the readers and invite submissions of that quality from various teachers. For the benefit of those members who could not attend the inaugural conference, there is an article by Shweta Naik and Ramanujam covering the events.

Given that there is a lot of misinformation regarding the precise historical contributions of ancient Indian mathematicians, it is crucial that informed experts write articles enlightening the mathematical community at large on these aspects. There is a beautiful, scholarly exposition of the Sulbasutras by Aditya Kolachana, K. Mahesh, and K. Ramasubramanian from IIT BOMBAY.

In the first issue, Amber Habib had described a very interesting method due to Bhaskara to approximate the sine function and illustrated how the visual representation of a function by its graph could be used to develop a strikingly accurate and easily computable approximation to the function. In the present issue, he continues to describe other historical contexts by reviewing how trigonometry is currently taught in Indian schools. He argues how certain episodes in the history of Indian mathematics provide opportunities for enrichment activities.

A panoramic view of the world of manifold infinities is described in a story-like fashion by Anindya Sen.

Apart from Ramanujan, the only other Indian mathematician whose name is likely to be known to the non-mathematician is Harish-Chandra. Even this cannot be said with confidence. One of the present-day eminent mathematicians Robert Langlands, had written a wonderful biography of Harish-Chandra for the Royal Society of UK. This describes in detail the areas entered into and contributions made to by Harish-Chandra. On behalf of the editors of Blackboard, A. Raghuram got in touch with Langlands who enthusiastically agreed to allow reprinting the article. On behalf of the editorial board, it is a pleasure to thank the editors of The Biographical Memoirs of Fellows of the Royal Society, especially John Coates and Helen Eaton, for granting us permission to reprint this biography of Harish-Chandra. The original article may be found on the website:

https://royalsocietypublishing.org/doi/10.1098/rsbm.1985.0008.

The first issue also contained some puzzles whose solutions were given at the end of the issue, and a set of mathematical problems which were left for members to try out. In this issue, we have given the solutions to these problems. As a matter of fact, MTA members will be encouraged to send in problems and articles based on their classroom experiences and other aspects of interest to teachers.

We had mentioned that the cover design is expected to change from issue to issue. However, from the feedback received, it appears that it would be a good idea to continue with the same design.

... *B. Sury, Indian Statistical Institute Bangalore.*

2 The Inaugural Conference of MTA - a Report *by Shweta Naik & R Ramanujam*

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R. Ramanujan, Institute of Mathematical Sciences Chennai.

The conference

Though MTA came into being last year and took up conducting Indian National Mathematics Olympiad rightaway, it was felt that a formal event inaugurating the Association was necessary, to reach out to the mathematics community in India and informing them of the new Association. Hence the conference, which was held at the Homi Bhabha Centre for Science Education, Mumbai in the first week of January this year. Indeed, *Blackboard* was launched at the conference.

The stated aim of the conference was to provide a forum for mathematics teachers, mathematics educators and mathematicians from all over the country to come together and discuss all aspects of mathematics education, and also to offer directions for the newly formed Association. We hoped that it would be a venue for discussions on mathematics teaching and learning at the school level as well as college level. Discussions on mathematics education would be on its curricular as well as pedagogic aspects.

The choice of January 3 as the date of inauguration was deliberate: it was the 188^{th} birth anniversary of the educational visionary *Savitribai Phule*. As early as in the late 19^{th} century, Jyotiba and Savitribai Phule saw universal education as the key to the upliftment of the oppressed classes and waged a valiant struggle for it. MTA is proud to associate itself with that memory right from its inception.

The programme

The association was formally inaugurated and its bulletin *Blackboard* launched by the eminent mathematician *M. S. Raghunathan* of IIT-Bombay (formerly from TIFR, Mumbai). In his keynote address, Raghunathan spoke on the state's responsibility to provide quality mathematics education for all, and the need to consider teachers as an important societal resource. Why should a good school teacher not earn as much as a university professor or an officer of the Indian Administrative Services and be respected as much, he asked. Indeed, she should.

Kishore Darak presented a biographical sketch of Savitribai Phule, and spoke on her struggle and convictions, and how it has inspired generations of Dalit educational activists in Maharashtra.

The conference programme included several theme sessions: on school mathematics, on college mathematics and on classroom experiences. In addition there were two half-day workshops held in parallel: one on **Technology in Math education** organized by *Jonaki Ghosh* of Lady Shri Ram College, Delhi, and the other on **Bridging school and college mathematics**, organized by *Geetha Venkataraman*, Ambedkar University, Delhi. While these looked at curriculum, pedagogy and classroom ethos of mathematics education, a panel discussion titled **Engaging every student in Mathematics: realising the vision of Savitribai Phule** brought policy issues and potential to our attention.

The conference's central function was fulfilled by contributed papers and posters presented by school and college teachers, mathematics education reserchers and educators. The event had a total of 11 paper presentations and 26 posters. Of these, 23 were by school teachers and the rest were educators representing NGOs across the country, SCERTs, Cluster resource centres, etc. Abstracts of the papers (and posters) can be found at **http://www.mtai.org.in/wp-content/uploads/2018/12/Binder1.pdf**. This issue of *Blackboard* highlights a few of them.

There was not much room for mathematics research at the inaugural conference, but there was a historical account of it, provided by India's pre-eminent number theorist *R. Balasubramanian* of the Institute of Mathematical Sciences, Chennai. He spoke on Number theory research in India after Srinivasa Ramanujan, in roughly the next 50 years. This was fascinating since few know of the excellent contributions to number theory from India in the decades just before and after independence.

A major highlight of the conference was a cultural event: **Haan, Mai Savitribai Phule** a solo play by *Sushma Deshpande*. Deshpande trained under the Augusto Boal, the Brazilian theatre practitioner, and learnt the techniques of the Theatre of the Oppressed. The play, originally written in Marathi as **Vhay, Mee Savitri Bai**, narrates the everyday life of Savitribai Phule, the wife of the 19th-century social reformer, Jyotirao Phule. In the play, Savitribai tells us the story of her life: her birth, marriage, life with Jyotirao Phule, their work for the upliftment of Dalits and marginalized women, the death of Jyotirao, and her own death from plague in 1897. A moving mono-act, the play touched and inspired everyone deeply.

The sessions

The theme session on **school mathematics** was organized by *R. Athmaraman* of the Association of Mathematics Teachers of India, and had presentations by

Anna Leena George (Dr. Dada Vaidya College of Education, Ponda, Goa), *S. R. Santhanam* (AMTI) and *Sneha Titus* (Azim Premji University, Bengaluru). Anna spoke of the fear of mathematics among students and illustrated mathematical experiences that inspired students instead. Santhanam emphasized the need for teachers' own understanding of mathematics. Sneha had thought provoking examples of mathematical explorations at school.

The theme session on **mathematics education at undergraduate level** was organized by *B. Sury* of the Indian Statistical Institute, Bengaluru. In it, *Amber Habib* (Shiv Nadar University, NCR), *Shobha Madan* (IIT Goa) and *Fozia Qazi* (IUST, Jammu and Kashmir) spoke. Amber spoke on the UGC curriculum and its rigidity. Shobha talked of the need for lectures that inspire students (and not merely educate). Fozia's talk was deeply moving, about the reality of teaching mathematics in a class where your students may not make it to class next day. It is amazing how much she and her university have accomplished yet.

The theme session on **classroom experiences** was organized by Shobha Madan and had presentations by *Sadiya Rehman* (Springdales School, Delhi), *Vinayak Sholapurkar* (S. P. College, Pune), *Kanchana Suryakumar* (Poorna Learning Centre, Bengaluru) and B. Sury. Sadiya spoke of her research on students' learning of mathematics relating it to her own interactions with students. Vinayak used the MTTS programme to highlight students' discovery of mathematics on their own, with teachers only guiding, rather than dictating the process. Kanchana gave us rare insights into mathematics assessment as a teaching / learning opportunity. Sury spoke of offering challenges that get students to think and explore.

The workshop on **Technology in mathematics education** began with an overview of technology tools available for exploring mathematical concepts, highlighting through examples their use in classrooms. This was followed by two sessions where participants got hands-on experience in exploring Dynamic Geometry Software (GeoGebra) and a spreadsheet (MS Excel/Calc). These sessions were conducted by Sangeeta Gulati (Sanskriti School, Delhi), Aaloka Kanhere (HBCSE, Mumbai) and Ajit Kumar (Institute of Chemical Technology, Mumbai).

The workshop on **Bridging school and college mathematics** had presentations by Shobha Bagai (Cluster Innovation Centre, Delhi), Anisa Chorwadwala (IISER, Pune), S. Kumaresan (Univ of Hyderabad (Retd)), Revathy Parameswaran (PS Senior Secondary School, Chennai) and Bhaba Kumar Sarma (IIT, Guwahati). Kumaresan kicked off the discussions with a fascinating account of the barriers faced by fresh undergraduate students, and how teachers could overcome these and indeed, help them take on mathematical challenges. Shobha used a linear algebra course as a scaffolding to illustrate the gap between school and college mathematics. Revathy presented a perspective from the school side, showing the tremendous challenges at higher secondary level, especially due to the Board exams. Bhaba gave us nuggets of learning from his extensive interactions with students and teachers.

The panel discussion on **Engaging every student in Mathematics** was organized by Bhaba Kumar Sarma and had Dinesh Lahoti (Edugenie, Guwahati), Anita Rampal (CIE, Delhi University) and Kameshwar Rao (NCERT (Retd)) as speakers. They all presented very different and important perspectives on universalisation of quality mathematics education. Anita spoke passionately on the regressive policy trends that alienate children and not engage them, contrasting with the successes of the NCERT *Mathemagic* books. Kameshwar Rao showed how gentle hints could bring the alienated child back into the discussion. Dinesh stayed outside the school, taking us on a tour of math melas, exhibitions and such, showing us their power.

With all these discussions, and a range of issues presented by contributed papers and posters, the inaugural conference had a rich discourse on mathematics education, showing us concern on its state in India, yet offering hope for the future.

Reflections

When the conference was being planned, there was some apprehension about response from teachers and participation from across the country. As we got closer to the conference date, we were instead overwhelmed by the scale of response. Regrettably we had to decline registration for a large number of mathematics teachers who wished to participate, only due to logistical reasons. While this positive signal from the community is heart warming for MTA, it underlines the need for regional conferences, so that large participation is facilitated.

If there is one area where the conference did not prove successful, it was in attracting the participation of non-English-speaking educators and teachers of mathematics. This is an area of concern and MTA will need to articulate and evolve policies that ensure discussions in Indian languages.

The Olympiad movement needed better representation in the conference as well. This is a growing section of mathematics teachers and students in the country, thirsting for challenges, and providing an immense opportunity for mathematical learning, MTA will need to harness this potential using its conferences.

Some teachers were disappointed that the conference did not provide them with resources they could take back to classrooms. We need meetings that fulfil different objectives and MTA will need to organize thematic conferences separately at school / college levels, and yet also find avenues that unite people across the spectrum. Use of online platforms for conferences was also suggested to expand the reach.

The inaugural conference has launched MTA well, the journey has begun. We look forward to building new relationships across the mathematics community in India.

3 बूझ बुझउअल - गणित सिखउअल Mathematical Folk Riddles of the Bhojpur Region *By Vijay A. Singh and Ranjana Pathak*

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Abstract

We present some mathematical examples from the rich tradition of folk riddles of the Bhojpur region. Several of these are in poetry form and can be sung. They represent a rare confluence of folk culture and mathematics which should be documented and encouraged. We point out possible links to the work of Aryabhata, Mahavira and others.

3.1 Introduction

Rural India has a rich oral tradition which can be traced to ancient times. Consider the shloka (Sanskrit verse):

यथा शिखा मयूराणां नागानां मणयो यथा । तथा वेदाङ्ग शास्त्राणां गणितं मूर्ध्नि स्थितम् ॥

(**वेदांग ज्योतिष** 500 BCE)

which means “Like the crowning crest of a peacock and the shining gem on the cobra’s hood, mathematics is the supreme *Vedanga Sastra*” . The six *Vedanga Sastras* are *Siksa* (phonetics), *Niruktam* (etymology), *Vyakaranam* (grammar), *Chandas* (prosody), *Kalpam* (ritualistics) and *Ganitam* (mathematics).

Not too long ago it was common for people in the Bhojpur region to sit by the fireside post dinner and pose riddles, often as poetry, some mathematical, but mostly non-mathematical. The mathematical riddles were called “Baithaki” which has a double meaning (i) people sit (*Baithak*) and solve; (ii) solve by guess-work, trial and error (*Baithana*). The study of the relationship between mathematics and culture is now termed Ethnomathematics.

The Bhojpur region spans eastern Uttar Pradesh and north-western Bihar. One of us (VAS) had some childhood knowledge of these riddles. However over the past decade and a half we have made conscious efforts to collate and classify them. In the next section we give three examples with two of them being in poetry form. Further, we connect them to the Indian mathematical tradition, specifically the work of Aryabhata (499 CE), Mahavira (circa 850 CE) and the Bakshali manuscript (probably 3rd century CE)

3.2 Three Examples

1) A True Baithaki Sawal.

Here is a riddle in poetry form which translates as

> *"I have 40 kg of iron/ I need to make 100 weapons/ The knife is a quarter the dagger one/ The sword in kg is five/ How many swords, daggers and knives?"*

In Bhojpuri the rhyme reads as follows:

मन भर लोहा सौ हथियार |
पउआ छूरी सेर कटार |
पच-पच सेर बने तलवार |
मन भर लोहा सौ हथियार ||

Setting it up in modern algebraic notation results in two equations in three variables. But this riddle is supposed to be an exercise undertaken by common folk sitting around a fireplace. So the idea is to solve it by trial and error. We have collected several riddles of this type, all in poetry form.

The interesting thing is that this tradition may perhaps owe something to Aryabhata who lived around this region some 1400 years ago. One could reduce the problem to linear Diophantine equations and then solve by Aryabhata's *Kuttaka* method. Another point to note is that such problems are topical and useful in current encryption standards.

2) A Tricky Algebra Problem.

This riddle, translated, reads as follows:

> *"In the fabled land of Mithila, the necklace of the darling daughter of king Birshbhan came apart. The pearls scattered... Her paramour stole a fifth, half fell to the floor. Thirty seven on her gown, sixty three on her bed. Seventy were stolen by her girl friends. So, how many pearls made up the necklace?"*

In mixed Bhojpiri-Maithli the rhyme reads as:

एक समय वृषभान दुलारी की
हार मिथिला में टूट गई रे।
अर्ध भाग भूमि पर गिरय्यो,
प्रिय पंचम भाग चुराई लियो रे।
सैंतीस अंचल, सेज तिरेसठ,
सत्तर सखियन ने लूट लियो रे।
कहो कितने मोतियन का हार भयो रे।।

This is an algebraic problem with a twist. The answer is 425 pearls. Once again this was to be solved as a "baithaki" problem by trial and error. We have tried it several times in workshops with school kids. Some came up with solutions and a few claimed to have sat up all night trying out various combinations until they hit upon the right number.

Another interesting point is that a similar problem can be found in the *Ganitha Sara Sangraha* of Mahavira, the 9th century Jain mathematician from Karnataka. It begins equally colorfully "One night in the spring season a charming young princess was meeting her paramour in a garden full of luxuriant flowers and fruits and resonant with the sound of koels and bees intoxicated with honey ..." The problem then goes on the describe how the princess' necklace broke and its beads were scattered. The fractions taken away by persons present are different from the above but the idea is similar.

3) A Thoughtless Proposal

This riddle, translated, reads as:

> *"A mendicant approaches the last driver of a sixteen cart caravan, each carrying 16 maunds of rice. The cart driver refuses his request for some rice saying that he should ask the driver ahead of him. Whatever he gets from him (the fifteenth), he (the sixteenth) will hand out double the amount. The mendicant goes from cart to cart and each driver brushes him off with the same lame sop that he will double the amount given by the driver immediately ahead. The first driver, taking pity on him, doles out some rice. Happily, the mendicant goes from driver to driver, diligently collecting double the amount. The last driver has to part with the entire cartload of sixteen maunds! So how much did the first driver give the mendicant?"*

The following conversions are provided:
1 maund = 40 ser (kg);
1 ser = 16 chatak = 80 tolas;
1 tola= 12 masa;
1 masa = 8 rati;
1 rati = 8 chawal;
1 chawal= 8 khaskhas.

[Note: Approximately, 1 tola = 11.5 gm and 1 chawal = 0.015 gm. Indeed, a short (not long) grain rice weighs about 0.015 gm.]

This is a problem on the addition of a geometric series and an appreciation of how, beginning with a small number one can arrive at a very, very large number. The sixteen maunds of rice have a weight of 39,321,600 chawals. Yet beginning with less than ten chawals (We deliberately do not give the precise answer here so that the reader may work this out) taken from the first cart the clever mendicant becomes master of a granary full of rice!

Several related problems can be found in Indian mathematical texts. The Bakhshali manuscript, probably from the 3rd century CE, has several such problems containing arithmetic and unusual series. A point to note is that Indians, even common folk, were perhaps comfortable with very large numbers.

3.3 Conclusion

We have tried to give a flavor of the the folk tradition of the Bhojpur region. It is interesting that puzzles have parallels with the works of Aryabhata and Mahavira separated as they are over centuries (in the case of Aryabhata) and also thousands of kilometers (Mahavira). As pointed out in the abstract, this tradition needs to be documented and nurtured. The advent of television, mass and electronic media has proved detrimental. We had to try hard to ferret out the riddles. They can be fun to pose and to solve.

Acknowledgement: One of us (VAS) acknowledges useful discussions with Prof. P P Divakaran, a fellow physicist and enthusiast of the Indian mathematical tradition.

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4 Explorations in Matchstick Geometry - A Classroom Experience *By Jayasree S*

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4.1 Introduction

In the article "*Needed: A Problem Solving Culture*" in the first issue of Blackboard, Professor Ramanujam talks about the need to provide opportunities for every child to engage in open mathematical explorations and expresses the hope that MTA-I would become a forum to share such activities. (Ramanujam, 2019) Following in this spirit, this article is an attempt to share one such exploratory mathematical activity and my experience of going through it with a group of grade IX students in a Corporation school in Chennai.[1]

One sure-fire way of generating a mathematical exploration is to ask the questions 'What if?' or 'What if not?' (Brown & Walter, 2005). In Euclidean geometry, we have access to two tools of construction namely straight- edge and compass. What if we had only the straight-edge? What if the only steps of construction allowed are laying unit-lengths (say match-sticks) end to end, including at an angle, without bending or breaking units? What sort of shapes and lengths are constructible under the 'restricted' rules of construction? What sort of geometry follows from these rules? How are the theorems of this geometry similar to or different from those of Euclidean Geometry? Questions such as these lead students on to explore the 'unfamiliar world' of matchstick geometry, made unfamiliar by adding a restriction to the familiar world of Euclidean Geometry.

The underlying rules of the world of matchstick shapes are different from those of Euclidean shapes and therefore the world that evolves is different as well. Let us look at some of the similarities and differences between these two worlds.

4.2 Matchstick world - Possibilities or otherwise

In Euclidean geometry, we have the straight-edge and compass construction to construct an equilateral triangle and therefore a 60° angle. Similarly, in matchstick geometry, we could arrange 3 unit lengths, meeting at the ends to form a

[1]I gratefully acknowledge the students who participated in the study, the mathematics teacher and the headmistress of the school who extended all support to make this possible

equilateral triangle and thus construct a 60° angle. How about a right angle? In Euclidean geometry, we have the construction for a perpendicular, which will give us a right angle. In matchstick geometry, we could align unit-lengths such that they form a right triangle with sides 3,4 and 5 units respectively to give us a right angle. We do not have a direct parallel for the straight-edge and compass construction of a perpendicular.

How about an angle of say 108°? One might be tempted to say ‘align five unit-lengths’ in such a way that they form a regular pentagon. But there are multiple ways of aligning the five unit lengths to form an equilateral pentagon, which are not necessarily equiangular as well and hence regular. So how does one get to that unique alignment where the angles are indeed 108°? Trial and error maybe – but that does not give us the steps of construction to construct a 108° angle. For the same reason, aligning four unit lengths to form a square would not be an acceptable set of steps of construction for a right angle, because one could align four unit lengths to form a rhombus as well. However the triangle with sides 3, 4 and 5 units is uniquely determined and hence definitely gives a right angle without involving trial and error or judgement. Thus it is clear that unlike in Euclidean geometry, where there is a sure-fire way to replicate any angle, in matchstick geometry only a restricted class of angles can be replicated. Clearly those angles that are parts of triangles with all three integer sides are replicable, which translates as those angles whose cosines are rational are replicable. Are these the only replicable angles in matchstick geometry? Some serious thinking is needed before one can answer this question.

There are other theorems like, say:

- Isosceles right triangles do not exist in matchstick geometry.
- Circles do not exist in matchstick geometry. (Obviously!)
- Irrational lengths cannot be constructed in matchstick geometry.
- In a rhombus whose angles are 60, 120, 60, 120 degrees, exactly one diagonal is constructible.
- Both the diagonals of a rhombus are constructible if and only if its side is of the form $n^2 + 1$.

These are within the reach of a secondary school student and are different from Euclidean geometry. However questions like “What are the interrelationships between these theorems?”, “Which ones can be derived from others and which are the fundamental assumptions?”, “What other theorems can be deduced from the ones we have?” etc. are questions that need further exploration.

4.3 The Classroom Implementation - Guiding thoughts

Several publications of the Association of Mathematics Teachers (ATM), UK, discuss implementations of such exploratory activities in the classroom. Termed *Investigations* in ATM literature, such activities involve presenting students with a mathematically rich situation, without a formally posed problem directing them to go one way or other. (Banwell, Saunders, & Tahta, 1986; Kissane, 1988; Yeo & Yeap, 2010) Following this tradition, we could have presented the activity with an open prompt such as 'Investigate matchstick shapes' However as (Mason, 1978) points out it is difficult to strike a balance between too much and too little guidance. Too little guidance produces paralysis because the students do not know how to proceed. Too much guidance induces them to solve the subsidiary tasks as individual tasks missing out the interrelationship between them, thus missing the larger picture.

We believe that the teacher has a crucial role to play in the exploration by designing appropriate triggers that stimulate thinking and generate further questions. She may also need to guide, encourage and prompt, if necessary directly, to keep the exploration going. An exploration does not happen by accident, but by design, not just of tasks, but of the way it unfolds in the classroom as well. The role of the teacher in this unfolding process cannot be overlooked.

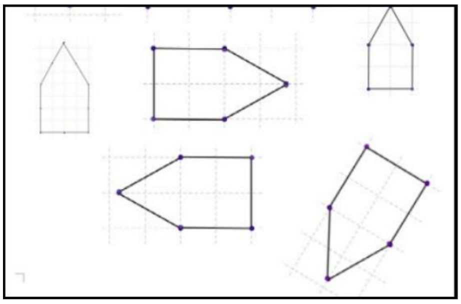
Given this belief, we structured the exploration such that it had three major sub-tasks, focussing respectively on
1) the notion of congruence and how it is similar or different in matchstick geometry,
2) describing shapes to enable identical replication and criteria for 'describability', and
3) constructibility of shapes.

The design and implementation of the tasks was such that students had sufficient opportunities to 'talk mathematics'. They were encouraged to work in groups, getting them to articulate their thoughts and prompting others to critique and build on them, leading to better clarity of the ideas involved for all concerned. Also, focussed worksheets were created on a couple of occasions to direct the discussion to some key issues, as described below.

We also thought that the hands-on experience of making matchstick shapes would be valuable in giving students a feel for what is possible and not possible and cue them to some of the differences discussed in the previous section. So the starting point of the exploratory activity involved students replicating matchstick shapes made by the teacher and making a few on their own. Three different unit lengths were used – matchsticks, toothpicks and broomsticks cut up to equal lengths.

4.4 From the Classroom

Task on notions of congruence: The use of three different unit-lengths turned out to be a lucky accident. This gave the space for questions as to whether the replicas made with different unit-lengths, with the same number of units along each side could be considered the same, to come up, bringing to the fore-front the notion of congruence. In order to focus the discussion a bit a worksheet including a) figures with the same number of unit-lengths per side, but of different unit-lengths b) scaled-up figures with different numbers of the same unit-lengths along the sides c) rotated and d) flipped figures was created. Students were asked to articulate their criteria for two shapes to be considered the same and to use their criteria to classify shapes in the worksheet as same or different. Some of the images from the worksheet are shown in the figure here.



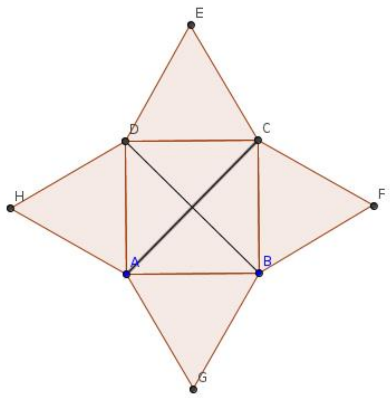
In the discussion that ensued, three different criteria for congruence emerged:
1) the usual notion of congruence as figures that can be superposed on one another,
2) a stronger version, that insisted on the number of unit lengths on each side being the same as well in addition to 1) above, and
3) a weaker version, which considered the equality of the number of unit-lengths on each side and the angles involved, but did not insist on superposition.

Questions were raised as to why one group should accept the definition of another and the need for one agreed upon definition. Given sufficient time, this discussion could have gone on to the choosing between these competing definitions, the criteria that could be adopted for the same and following through to some of the consequences/theorems that could be derived from each of these. For example with the usual notion of congruence the area of congruent figures remains invariant. Would this invariance still hold under definition 2) or 3) above?

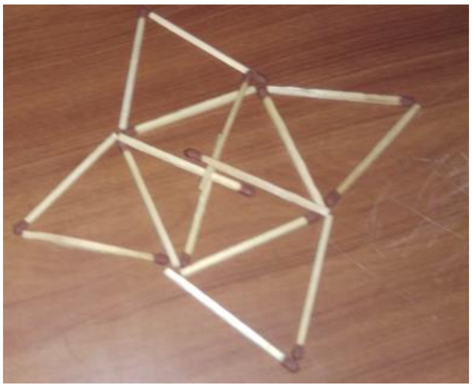
Task on describing shapes provided opportunities for students to use mathematical vocabulary. In this task, one student from each group was shown a shape and asked to describe it to the others, as if on a telephone, so as to enable them to replicate the shape. This task saw the evolution of vocabulary from loose descriptions like 'a small square' to more precise use of mathematical language. In trying to classify a set of figures as describable or otherwise, students had a hunch that it had something to do with the angles present in the figure.

We did not go further on this activity, but as in the earlier example, this also leaves open many questions to pursue – For example, how are the angles present in a shape related to its describability? How can one prove (or disprove!) the relationship? Can we come up with a set of criteria for describable shapes? Can we look for a minimal description – say one that specifies the minimum essential conditions – to uniquely describe a shape? Some of these are non-trivial questions which can keep one occupied for days.

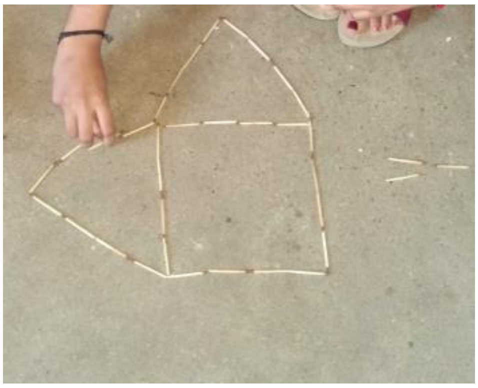
Task on constructibility of shapes: In this task students were asked to replicate the figure below.



After the failure of the initial attempts they argued that it is not possible with one matchstick per side – as the diagonal of a square being longer than the side, will have to have more matchsticks than those along the sides. With a unit square, a diagonal one-unit long is too short and two units long is too long. So the sticks will have to overlap.



They then moved on to exploring if it is possible with larger squares – for example it may be possible that a 3 unit-long square could have a 4-unit diagonal.



Soon some students in the class had a strong hunch that it would not be possible to make a diagonal for any square. Some others claimed that it is possible by making a square that was 2-units long and a reasonably well-fitted diagonal of length 3 units. Now the class was in a dilemma. One group in the class felt that there was something wrong with the square: They said, they need to ensure that the matchsticks are indeed of the same length. Some said the thickness of the matchstick might interfere and wanted to make a pencil drawing of the same. In the meantime, some of them drew on known mathematics to justify that there is indeed something wrong with the 2 unit square having a diagonal of length 3 units. They argued that, if this were so, according to Pythagoras Theorem, the square of the diagonal ought to be $2^2 + 2^2$, which is not equal to 3^2. The length of the diagonal has to be more than 2, but less than 3 and so is not constructible in matchstick geometry.

The students now asked if it is possible to construct the diagonal of any matchstick square at all. They found that the square of the length of the diagonal comes to twice a perfect square , numbers such as 8, 18, 32, 50, etc and none of these have an integer square-root. They were generalising from examples and implicitly using the irrationality of root 2. With some help from the teacher this was proved as well. Thus in this task, we saw students moving from generalising from observed reality to using known mathematics to justify their conclusion and further on to generalising the conclusion and to proving it with some support.

Further questions about constructibility of diagonals of other shapes like rectangles or rhombuses, or altitudes of triangles can be raised. In general, this gives rise to the question, what shapes are constructible if only integer lengths are allowed. If we impose the additional condition that the distance between any two vertices (instead of only pairs of adjacent vertices in matchstick geometry) has to be an integer, we are bang on to integer geometry, where the possibilities for further explorations are endless.

4.5 Conclusion

In the above examples, we tried to provide some possible directions to explore matchstick geometry. Each of the three tasks described, can be extended much further as can be seen from the further questions raised in each case. Some of these may be appropriate for even undergraduate students. At the same time, there is something for each child to work on, progress and experience the joy of finding something out for herself. This can be a motivation for many to take to mathematics or at best to mitigate the fear that they have of it. These explorations provide them with ample opportunities to pose and solve problems, generate examples and counterexamples, come up with notations, representations and definitions, make and test conjectures, generalise and specialise, justify generalisations, and communicate mathematics to their peers. Provided an opportunity, students do engage in these processes enthusiastically and come up with questions of their own to explore. Whether all answers to the questions posed are arrived at or not, it is important that the questions be raised and options explored. This gives students a taste of what it means to 'do' mathematics.

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5 Mathematics and the Sustainable Development Goals: Interdisciplinary learning through a digital game *By Robin Sharma*

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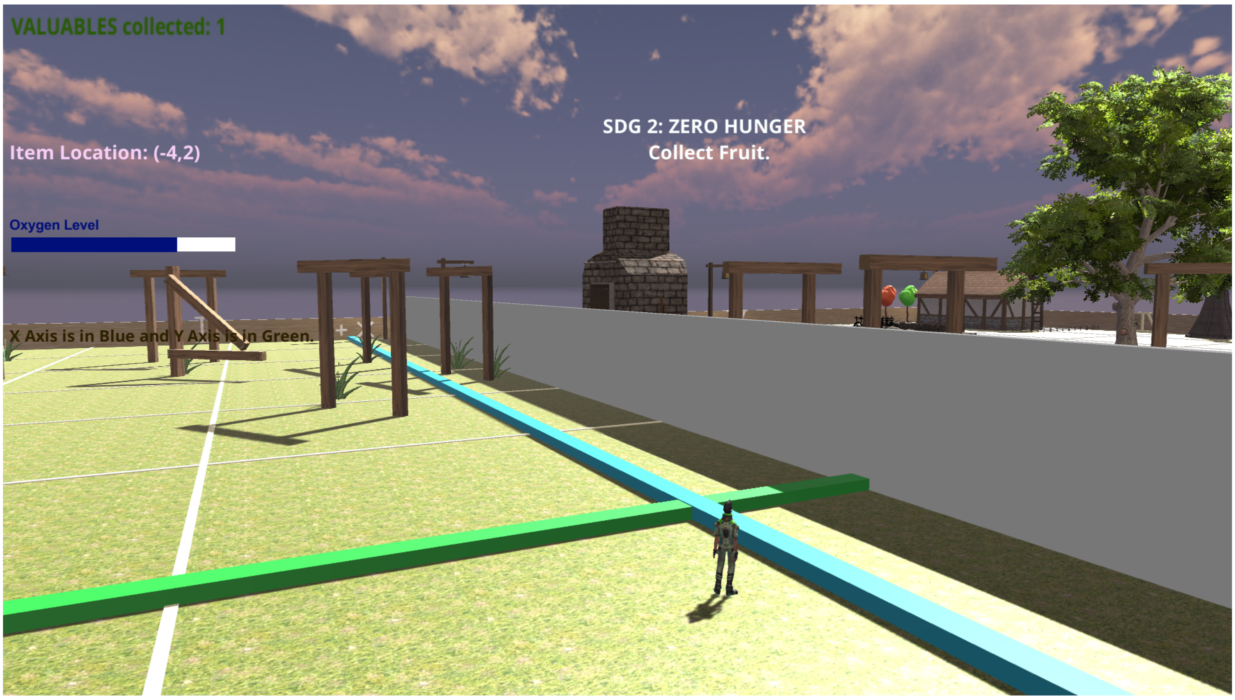
Back in 2005, the NCF[1] envisioned school mathematics to take place in a situation where "children learn to enjoy mathematics" and it "becomes a part of their life experience which they talk about". It recommended "enriching teachers with a variety of mathematical resources" as one of the ways to achieving this vision. But what kind of resources are we talking about? Who will develop these resources? How will we be able to make these resources available to the teachers? These are some questions that the mathematics education community is still grappling with, even after 14 years of the visionary document.

While we have made significant progress in providing ICT infrastructure to students (mostly urban), there is still dearth of "quality" content. Either the computers or smart boards are eating dust or are merely used to project PDF versions of textbooks and/or display boring Presentations. We are still operating in the "transmissive" model of merely delivering content with no student agency or interactivity. Offering multi-modal learning content which is immersive, engaging and interactive should be at the heart of learner-centric pedagogy. This is where ICT is promising and can play a crucial role. With the advent of low-cost, high-speed connectivity and diminishing administrative resistance, ICTs are making way into classrooms. At the same time, we have to be cognizant of the fact that the responsibility of effective usage of infrastructure and design and development of new ICT based content cannot be entirely tossed on the teachers. I believe we need a totally different group of individuals who provide support and work with teachers in designing and developing contextualized content for the 21st century learner. Teacher advisers, curriculum and content developers should work with teachers to develop content which then the teacher can deliver in the classroom.

In this article, I present an example of a learning resource, a digital game ti-

[1]National Curriculum Framework (2005), National Council of Educational Research and Training

tled ‘Dimension Destination’. The windows-computer based game incorporated feedback from students, teachers, game designers and researchers in its design. It was learnt that while interdisciplinarity has become relevant for emergent sciences and social sciences of the 21st century, the discipline of mathematics is still rooted in rudimentary computational processes and problems[2]. Designed for secondary grade students, the game incorporates a narrative around the SDGs[3] while providing opportunity to the learner (player) to apply her coordinate geometry skills. In addition, the idea is to also inculcate spatial geometry skills in the player. The game’s interface is imprinted with the Cartesian Axes system and provides the player with coordinates of collectibles associated with different SDGs that she must collect to be able to move forward. The game takes place in the year 2030, humans have not been able to achieve the SDGs, the player finds herself on an unknown deserted extra-terrestrial object and she must collect various valuables to help Earth achieve the SDGs. As she collects a valuable, she is provided with coordinates of the next and she must use her coordinate geometry skills to find each valuable before her Oxygen runs out.



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Screenshots from the Game

Game based learning has direct benefits from learners' point of view. It provides learner agency and helps the learner take control of her own learning. Digital game based learning, if used and implemented efficaciously has huge potential for the multi-modal, scale-ability and personalization freedom it can provide. Preliminary results from using Dimension Destination in classrooms have been encouraging. Students were engaged and enjoyed playing the game. It is to be noted that digital games shall not be seen as independent resources. Discussions and blended learning methodologies should ideally complement gameplay in the classroom.

Here are some quotes from students after playing Dimension Destination for a short-session.

"I could see how the latitude and longitude system that we have is a direct application of coordinate geometry and the Cartesian Coordinate System."

"I didn't anything about the SDGs. I think I would probably go back, read more about them and see how we can make them a reality for the well-being of our planet."

"The fact that the GPS (Global Positioning System) that we use on our mobile phones to find routes is based on coordinate geometry just hit me after playing the game."

Designed for Windows using Unity 3D, the game discussed in this article is only a model developed for research purposes and is currently not available widespread use and deployment. For using the game in your classroom, reach the author at: r.sharma@unesco.org

6 History in the Classroom: Approximating the Sine Function II *By Amber Habib*

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Abstract

In this note, we review how trigonometry is currently taught in Indian schools. This reveals several opportunities for enrichment activities based on episodes in the history of Indian mathematics. In particular, we can employ the sine table of Varahamihira and the quadratic interpolation formula of Brahmagupta.

6.1 Introduction

In the first part of this article [1], we had made a case for using episodes from history to present different perspectives on a topic and thus to show connections between parts of mathematics. We used Bhaskara's approximation of the sine function to illustrate how the visual representation of a function by its graph could be used to develop a strikingly accurate and easy to compute approximation to the function.

In this sequel, we begin by reviewing the presentation of trigonometry in the NCERT textbooks. We compare this presentation with the 2006 recommendations of the National Focus Group on Teaching of Mathematics [2] and make suggestions for how history could be brought into the classroom to implement certain aspects of these recommendations.

6.2 Trigonometry in School

Under the current school syllabus as captured by the NCERT books, the study of trigonometric ratios happens essentially in classes X and XI. The Class X textbook [3] begins, in Chapter 8, with stating a few 'real-life' situations where trigonometry could be used to measure a distance, such as one involving a drifting balloon.

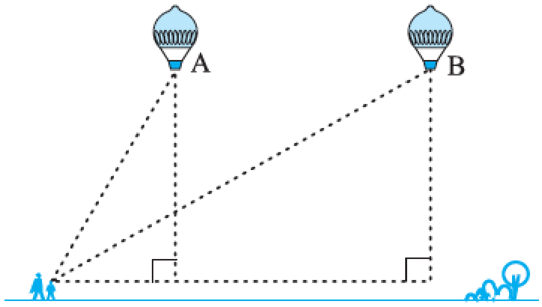


Figure 1: The balloon has drifted horizontally. Can we find out its height? (Figure 8.3 in [3])

It then introduces the six trigonometric ratios for right-angled triangles together with their values for the standard angles $\theta = 0$, 30, 45, 60 and 90 degrees. The identity $\sin^2\theta + \cos^2\theta = 1$ and its consequences $\tan^2\theta + 1 = \sec^2\theta$, $\cot^2\theta + 1 = \csc^2\theta$ are proved. The student is expected to be able to find the trigonometric ratios from given lengths of sides, to assess expressions and equalities involving the trigonometric ratios of the standard angles, and to justify new identities.

Subsequently, in Chapter 9, the book returns to the motivating problems of applying trigonometry to problems such as finding the width of a river or the height of a balloon.

Moving on to the Class XI textbook [4], we are introduced to the unit circle picture and to the concept of angles whose measure extends over all real numbers. The trigonometric ratios now become trigonometric functions, we observe their periodicity, and we obtain the standard identities such as $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ and $\cos\theta + \cos\phi = 2\cos(\dfrac{\theta+\phi}{2})\cos(\dfrac{\theta-\phi}{2})$. The known values of trigonometric functions are slightly extended, to cover 180, 270 and 360 degrees. The applications of this new knowledge are confined to derivations of further identities, and to the calculation of the ratios for a few more angles such as 15°, 22.5°, 75° and 195°.

The fundamental problem of calculating the trigonometric values for arbitrary angles is not stated, let alone emphasized. Why do we develop identities? Apparently, to develop still other identities and to solve some trigonometric equations. Where do the trigonometric equations come from? They are just there. The problems on applications are also stunted by necessarily being based on angles of 30, 45 and 60 degrees. A student trying to actually apply trigonometry outside the classroom will not get very far. The treatment cries out for structure and purpose.

The National Focus Group had suggested a shift in focus from content to processes: "formal problem solving, use of heuristics, estimation and approximation, optimisation, use of patterns, visualisation, representation, reasoning and proof, making connections, mathematical communication." The remaining part of this article is devoted to how episodes from history can serve to illuminate some of these aspects that have currently got short shrift. First, the scattered development of identities can be given a purpose: to calculate the trigonometric ratios for equally spaced angles with small gaps. This will give the study an overarching structure. Second, we can learn how to estimate the ratios for arbitrary angles. This is essential for practical applications. Along the way, we also learn how to get accurate approximations of π.

6.3 Varahamihira's Sine Table

Given that our purpose is to enliven the classroom, we will use current terminology to present our ancestors' work. This may obscure some of the subtleties or originality of their work. We begin with Varahamihira's *Pancha-Siddhantika* [5] where he surveys the work of previous generations from the vantage point of the 6th century CE.

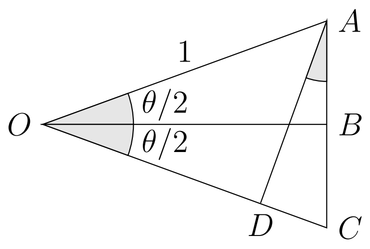
The *Pancha-Siddhantika* explains that a trigonometric table can be constructed using the identities

$$\sin^2 \theta/2 = \frac{1-\cos\theta}{2} \tag{1}$$

$$\cos^2\theta = 1 - \sin^2\theta \tag{2}$$

$$\cos\theta = \sin(90^\circ - \theta) \tag{3}$$

The second identity is of course fundamental to the trigonometry taught in Class X, while the third is just the definition of cosine. Though the justification of the first must also be familiar to the reader, we still go over it to emphasize that it is well within the reach of a Class X student.



First, we draw the right-angled $\triangle OBA$ with the angle at O being $\theta/2$ and the hypotenuse OA having unit length. Then we draw its reflection, $\triangle OBC$. We drop a perpendicular from A to OC, meeting OC at D. Observe that $\triangle OBA$ and $\triangle OBC$ are congruent to each other and that $\angle DAC = \theta/2$.

We have $|AC| = 2|AB| = 2\sin\theta/2$, $|AD| = \sin\theta$ and $|DC| = |OC| - |OD| = 1 - \cos\theta$. Applying the Pythagoras theorem to $\triangle ADC$, we get

$$4\sin^2\theta/2 = \sin^2\theta + (1-\cos\theta)^2$$

Substituting $1 - \cos^2\theta$ for $\sin^2\theta$ gives (1).

Now we can implement Varahamihira's prescription for constructing a table of sines and cosines. We begin with the known values for the angles of 30, 45 and 60 degrees, leaving out the trivial cases of 0 and 90 degrees.

- Put $\theta = 30°$ in (1) to find $\sin 15°$. Then put $\theta = 15°$ in (2) to get $\cos 15°$. Again, put $\theta = 15°$ and $75°$ in (2) to get $\sin(75°)$ and $\cos(75°)$. At this stage we have the sines and cosines of the following angles: 15, 30, 45, 60 and 75 degrees. (For the record, $\sin 15° = \frac{1}{2}\sqrt{2-\sqrt{3}} \approx 0.259$ and $\sin 75° = \frac{1}{2}\sqrt{2+\sqrt{3}} \approx 0.966$.)

- Use (1) to find the sines of the following half-angles: $7.5°$, $22.5°$ and $37.5°$. Then use (2) to find the corresponding cosines. Finally, use (3) to find the sines and cosines of the complementary angles $52.5°$, $67.5°$ and $82.5°$. We now have all the sines and cosines at increments of $7.5°$.

- Again use (1) to get the values for the further half-angles such as $3.75°$, (2) for the corresponding cosines, and (3) for the sines and cosines of the complementary angles, to get a table with increments of $3.75°$.

Varahamihira stopped at this stage. We may decide not to go even so far. Just adding 15 and 75 degrees to the table allows significantly greater variety in the exercises. On the other hand, this is a suitable activity for the mathematics or computer lab.

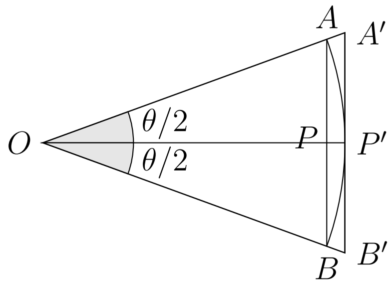
Activity 6.1. *Implement Varahamihira's algorithm in a spreadsheet program like Microsoft Excel or LibreOffice Calc.*

(i) Compare your table with the one Varahamihira gave [6, p. 51], keeping in mind that his 'Sine' is actually the height of a right-angled triangle with hypotenuse 120. Further the lengths are in base 60. A Sine of 7;51 represents $7\frac{51}{60} = 7.85$ *and the corresponding sine (height/hypotenuse) is* $7.85/120 = 0.0654$.

(ii) Extend the table twice more, to angles which are multiples of $0.9375°$ *or* $3375''$.

6.4 Approximating π

One can also focus on using Varahamihira's process to get sines of successively smaller angles starting at $30°$ and then proceeding through $15°$, $7.5°$, $3.75°$, $1.875°$, and so on. These can be used to get bounds for π. At present students learn that 22/7 and 3.14 can be used for π in numerical problems. Unaware of how such estimates are arrived at, many are left with the impression that 22/7 and 3.14 *are* π, ignoring the fact that they are not even equal to each other.



We observe that π is the area of a circle with radius one. Let us consider two concentric regular polygons with n sides, one inscribed in this circle and one circumscribing it. Let each side of the polygon subtend an angle θ at the centre. The diagram on the left depicts a part of this situation. AB is a side of the inscribed polygon and $A'B'$ is a side of the circumscribing one.

We have $|OA| = |OP'| = |OB| = 1$. We also have

$$\text{Area}(\triangle OAB) = \cos\theta/2 \sin\theta/2 = \frac{1}{2}\sin\theta \quad \text{and} \quad \text{Area}(\triangle OA'B') = \tan\theta/2.$$

If $\theta = 360°/n$, each regular polygon has n such triangles, giving the following bounds for π:

$$\frac{n}{2}\sin\theta < \pi < n\tan\frac{\theta}{2} \tag{4}$$

For example, $\theta = 30°$ gives $6\sin 30 < \pi < 12\tan 15°$, or $3 < \pi < 3.22$. Working with smaller θ will give tighter bounds for π and this is also best done on a computer, say with Excel or Calc. Students could try to answer the following:

Activity 6.2. *Use Varahamihira's scheme to find the trigonometric ratios for angles that are successively halved, starting with* $\theta = 30°$ *and then covering* $\theta = 15°$, $7.5°$, $3.75°$, $1.875°$ *etc. Use (4) to get corresponding bounds for* π.

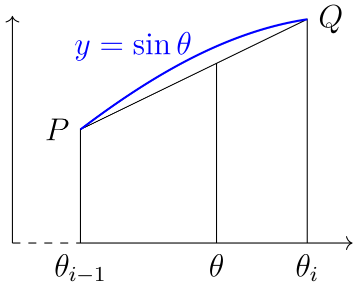
(i) Which choice of $\theta = 360°/n$ *shows that* 3.14 *is an underestimate of* π*?*

(ii) Which choice of $\theta = 360°/n$ *shows that* 22/7 *is an overestimate of* π*?*

(iii) Aryabhata gave an estimate $\pi \approx 3.1416$ *in the 5th century CE. A regular polygon of how many sides would be needed for that level of accuracy?*

6.5 Brahmagupta's Quadratic Interpolation

The presentation in this section is based on graphs of functions and so it is best suited to students in Class XI or XII. As we have seen, the typical Indian trigonometric table tabulated values at intervals of $\delta = 3.75°$. When sines of other angles were required, they were estimated by *linear interpolation.*



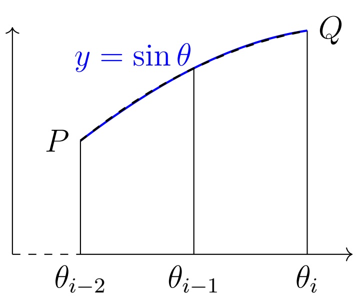
In linear interpolation, we first match the graph of the function $\sin\theta$ with a straight line which meets it at the points P and Q corresponding to successive angles θ_{i-1} and θ_i in the table, with $\theta_{i-1} < \theta < \theta_i$. To estimate $\sin\theta$ we take the point on PQ which lies directly above θ. This gives:

$$\begin{aligned}\sin\theta &\approx \sin\theta_{i-1} + \frac{\theta - \theta_{i-1}}{\theta_i - \theta_{i-1}}(\sin\theta_i - \sin\theta_{i-1}) \\ &= \sin\theta_{i-1} + \frac{\theta - \theta_{i-1}}{\delta}\triangle\sin\theta_i. \qquad (5)\end{aligned}$$

Note 6.3. *Given a sequence of numbers a_1, a_2, ..., their* first differences *are a new sequence $\triangle a_1$, $\triangle a_2$,...defined by $\triangle a_i = a_i - a_{i-1}$ and with the convention $a_0 = 0$. Their* second differences *are then defined by $\triangle^2 a_i = \triangle a_i - \triangle a_{i-1}$.*

Activity 6.4. *Estimate* $\sin 80°$ *using linear interpolation and the known values of* $\sin 75°$ *and* $\sin 90°$*. Compare with the value of* $\sin 80°$ *provided by a scientific calculator. What is the relative error? Similarly estimate the relative error at a few other points.*

For greater accuracy of the interpolation, one could first create a table with more closely packed angles. In Egypt, during the 2nd century CE, the Greek mathematician Ptolemy had already managed to tabulate chords at half degree intervals and this was equivalent to tabulating sines at 0.25° intervals! In India, however, the tables continued for some centuries to maintain the much larger gap of 3.75°. Finally, in the 7th century CE, Brahmagupta offered a radically different approach, replacing linear interpolation with quadratic interpolation.



In quadratic interpolation, we match the graph of $\sin\theta$ with a parabola which meets it at 3 points, corresponding to θ_{i-2}, θ_{i-1} and θ_i. The diagram on the left shows the improvement, the blue curve being the sine function and the dashed curve being the quadratic interpolation. As with Bhaskara's approach [1], the approximation is near perfect.

We do not know how Brahmagupta arrived at his formula. We present below a derivation that is in the same spirit as an explanation given by Bhaskara II in the 12th century CE [7, pp. 111-12]. We start with a form similar to the linear interpolation formula:

$$\sin\theta \approx \sin\theta_{i-1} + (\theta - \theta_{i-1})\, p(\theta)$$

where $p(\theta)$ is a linear function $A\theta + B$. This already does the right thing at θ_{i-1}. We have to choose $p(\theta)$ so that it also does the right thing at θ_{i-2} and θ_i. In other words, we need $p(\theta)$ to satisfy the following:

$$\begin{aligned}\sin\theta_{i-2} &= \sin\theta_{i-1} + (\theta_{i-2} - \theta_{i-1})p(\theta_{i-2}),\\ \sin\theta_i &= \sin\theta_{i-1} + (\theta_i - \theta_{i-1})p(\theta_i).\end{aligned}$$

These can be rearranged into

$$p(\theta_{i-2}) = \frac{\triangle\sin\theta_{i-1}}{\delta} \quad \text{and} \quad p(\theta_i) = \frac{\triangle\sin\theta_i}{\delta}.$$

This is again a linear interpolation problem and we have already seen how to solve it:

$$\begin{aligned}p(\theta) &= p(\theta_{i-2}) + \frac{\theta - \theta_{i-2}}{\theta_i - \theta_{i-2}}\,(p(\theta_i) - p(\theta_{i-2}))\\ &= \frac{\triangle\sin\theta_{i-1}}{\delta} + \frac{\theta - \theta_{i-2}}{2\delta}\left(\frac{\triangle\sin\theta_i - \triangle\sin\theta_{i-1}}{\delta}\right)\\ &= \frac{\triangle\sin\theta_{i-1}}{\delta} + \frac{\theta - \theta_{i-1} + \delta}{2\delta}\left(\frac{\triangle\sin\theta_i - \triangle\sin\theta_{i-1}}{\delta}\right)\\ &= \frac{\triangle\sin\theta_{i-1} + \triangle\sin\theta_i}{2\delta} + \frac{\theta - \theta_{i-1}}{\delta^2}\left(\frac{\triangle^2\sin\theta_i}{2}\right)\end{aligned}$$

We bring things to a close by putting this expression for $p(\theta)$ back into our initial quadratic approximation. We get the formula given by Brahmagupta:

$$\begin{aligned}\sin\theta \approx \sin\theta_{i-1} + \frac{\theta - \theta_{i-1}}{\delta}\Bigg(&\frac{\triangle\sin\theta_{i-1} + \triangle\sin\theta_i}{2}\\ &+\frac{\theta - \theta_{i-1}}{\delta}\left(\frac{\triangle^2\sin\theta_i}{2}\right)\Bigg)\end{aligned} \tag{6}$$

for $\theta_{i-2} < \theta < \theta_i$.

Brahmagupta used $\delta = 15°$. So he required a trigonometric table with only five angles (ignoring the trivial cases of 0 and 90 degrees): 15, 30, 45, 60 and 75 degrees.

We should note that Brahmagupta's formula does not depend on any specific properties of the sine function and so it can be used for *any* function.

Activity 6.5. *Estimate* $\sin 80°$ *using the quadratic interpolation formula (6) and based on a table with* $\delta = 15°$*. Compare with the value of* $\sin 80°$ *provided by a scientific calculator. What is the relative error? Similarly estimate the relative error at a few other points. Compare the accuracy with linear interpolation using a table with* $\delta = 3.75°$*.*

In this article, we have concentrated on the mathematics and have not included details of our protagonists and their setting. For such information, which is quite essential to the classroom, see [6, 7].

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7 The Ladder of Infinity *by Anindya Sen*

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Infinity comes in many different sizes.

Yes, it's true. Get over it.
Or at least, let's think about why it is so difficult to get over it.
After all, nobody is particularly bothered when I say "Numbers come in many sizes".

I suspect the reason is that when people hear "Infinity" they think of the **Poetic Definition:** Infinity is the Ultimate Maximum.

This notion of Infinity as the Absolute, the Whole, "Something greater than which nothing can be conceived" has a long history in human culture.
Notions of the Infinite are frequently associated with a Supreme Being.

Viewed in this light, the idea of different sizes of infinity is indeed very confusing.
How can there be a larger or smaller Ultimate Maximum?

The scientific study of infinity – a branch of mathematics known as Set Theory – begins by descending from the lofty heights of the Absolute and starts with a **Prosaic Definition:** An infinite set is a collection of objects that does not end.

Viewed in this light, different sizes of infinity suddenly look more plausible.
For instance, consider the collections $\{0, 1, 2, 3, 4, 5 \ldots\}$ and $\{2, 3, 4, 5 \ldots\}$.
Both never end, but the latter has two fewer members than the former.

Now if you look at $\{3, 5, 7, 9 \ldots\}$ (the odd numbers greater than 1), this has infinitely fewer members than both the above.
Surely it is conceivable that, while all three sets above are infinite, they have different sizes – just like the numbers 5, 17 and 5000 are all bigger than 2, but different from each other?

But how can we be sure?
After all, as everyone knows, only Rajinikanth can count to infinity (twice). ☺
Let's start at the very beginning – with the idea of a Set.

The Joy of Sets

A set is a collection of objects. (Some technical conditions apply but they can wait.)
We take the objects belonging to the collection and put curly brackets around them.
$\{1, 2, 3\}$ is a set. So is $\{a, b, c\}$ or $\{\text{Jack, Jill}\}$.

The objects belonging to a set are called its members.
1 is a member of $\{1, 2, 3\}$ and b is a member of $\{a, b, c\}$, but 1 is not a member of $\{a, b, c\}$ and vice versa.

Of particular interest to us will be **N**, the set of positive integers (called "Natural Numbers" by mathematicians).
N= $\{0, 1, 2, 3, 4 \ldots\}$ is an example of an infinite set, as opposed to the other sets we saw above which are finite.

Once you have a few sets in hand, you can more sets out of them, using a couple of operations.

Union: Given two sets A and B, their union $A \cup B$ consists of all the members that belong to A or to B.
So, if $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, then $A \cup B = \{1, 2, 3, 4, 6\}$.

Using this idea repeatedly, you can define the union of any number of sets or even an infinite number of sets.
So, for instance, if $A_1 = \{0\}$, $A_2 = \{0,\ 1\}$, $A_3 = \{0,\ 1,\ 2\}$ and so on, we can take the union of all the sets to get $A_1 \cup A_2 \cup A_3 \cup \cdots = \{0, 1, 2, 3, 4 \ldots\} =$ Our old friend, **N**.

Intersection: Given two sets A and B, their intersection $A \cap B$ consists of all the members which belong to *both* A and B.
So, if $A = \{1,\ 2,\ 3\}$ and $B = \{2,\ 4,\ 6\}$, then $A \cap B = \{2\}$.

What if A and B have *nothing* in common? Let's say if $A = \{1,\ 2,\ 3\}$ and $B = \{4,\ 5,\ 6\}$?
Well, in that case, $A \cap B$ is defined to be the *Empty Set*, which contains nothing at all.
The Empty Set is denoted by { }, (there's nothing inside the brackets) or the letter $\emptyset$.

Just like unions, you can also talk about the intersection of infinitely many sets, although we won't make much use of them in this post.

***Subsets*:** A set A is said to be *a subset* of another set B, if *every member* of the set A also belongs to B.
So for instance $\{2\}$ and $\{2, 6\}$ are both subsets of $B = \{2, 4, 6\}$.
But $\{2, 5\}$ is not.

Two important things to note:
– Any set is a subset of itself. So $\{2, 4, 6\}$ is a subset of $\{2, 4, 6\}$. (Check that the definition is satisfied!)
– The Empty Set, { }, is always a subset of every set.

These last two facts often confuse newcomers to set theory, but here's a way to think about it.
To generate a subset of a given set – go through every member of the set, one by one, and make a decision about whether to include that member or not.
Every possible set of decisions corresponds to a valid subset.

So, if $A = \{3,\ 4,\ 5\}$, you could decide to include only the first member and exclude the rest.
This gives you the subset $\{3\}$.
If you decide to include the first two and exclude the third, you get $\{3, 4\}$.
If you decide to include *all* the members, that gives you the set A.
If you decide to include *none*, you get the empty set.

***Exercise 1*:** If a set, A, has n members, show that there are 2^n possible subsets of A.

With subsets under our belt, we come to a crucial concept for studying infinity.

Firstly, note that *a set can have other sets as its members.*

So, for instance: A = {{1, 2}, {3, 4, 5}, {a, b}} is a set.
Note that {1, 2} and {a,b} are *members* of A. But {{1,2}, {a,b}} is a *subset* of A.
However, {1, 2, 5} is *not a member* of A. (Neither is it a subset)

Now, are ready to define:
***Power Set*:** The power set of a set$A$, is the set $P(A)$, consisting of *all* subsets of A.
Best to illustrate this by example:
If $A = \{1,\ 2,\ 3\}$, the power set of A is given by:
$P(A)$ = { { }, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} } .
(Note carefully the placement of curly brackets and commas)

***Exercise 2*:**
(easy) If A has n members, show that $P(A)$ has 2^n members.
(harder) If $A = \{1,\ 2\}$, write down $P(P(A))$ ("power set of power set of A").

Note that everything we are doing can be defined for an infinite set like **N**.
The power set of **N**will consist of finite subsets like {3, 5, 7} and well as infinite subsets like
All the odd numbers

So, what's all this got to do with different sizes of infinity?
Well, the language of sets allows us to define sizes of sets – finite or infinite – in a very precise way.
In math jargon, the word used for "size" is "cardinality", but we'll stick with "size".

Sizing Things Up

Imagine you have forgotten how to count.

Is there any way for you to distinguish the size of two different sets?
For instance, is there any way for you to recognize and convey the fact that {1, 2, 3} has just as many members as {a, b, c}, but fewer members than {a, b, c, d, e}?

Turns out that there is.
In fact, this is how pretty much everybody learns to count but then forgets!
It's the very basic idea of pairing things up.

The idea is, given two sets A and B, I will try to pair up every member of A with some member of B – with the stipulation that different members of A must be paired with different members of B.
(In fancy math-speak, a pairing as above is called a *one-to-one function*)

In what follows, I will use the notation "$x \to y$" as shorthand for "x is paired with y".

So, if $A = \{1,\ 2,\ 3\}$ and $B = \{a,\ b,\ c\}$, then: $1 \to a$, $2 \to b$, $3 \to c$ is a valid pairing.
So is: $1 \to b$, $2 \to c$, $3 \to a$.
But: $1 \to a$, $2 \to a$, $3 \to b$ is invalid because 1 and 2 are both paired with a.

But if we took, $B = \{\text{a, b, c, d, e}\}$ instead, then $1 \to a$, $2 \to b$, $3 \to c$ or $1 \to b$, $2 \to d$, $3 \to e$ are both valid ways of pairing all members of A with some members of B (there are many other ways).

The crucial difference, is that, every member of $B = \{a,\ b,\ c\}$ ended up getting paired with some member of $\{1,\ 2,\ 3\}$ but when we took $B = \{\text{a, b, c, d, e}\}$ instead, then however hard you try to pair things up, you always end up missing some members of B.

Hence, even without knowing how to count, one can compare sizes of sets, in a way which matches perfectly with our intuition of one set having "more things in it" than another one.
Let's make this official.

***Definition 1*:** Given two sets A and B, we say that $A \leq B$ (Size of A is less than or equal to size of B) if there is *some way* of pairing up *all members* of A with *some members* of B (but not necessarily all of them).
In math jargon, this means there is an *injection* from the set A *into* the set B.

***Definition 2*:** Given two sets A and B, we say $A = B$ (A and B have the same size), if there is *some way* of pairing up *all members* of A with *all members* of B.

In math jargon, this means there is a *bijection* from the set A *onto* the set B.

***Definition 3*:** Given two sets A and B, we say $A < B$ (A is smaller than B or B is greater than A) if:
– There is *some way* of pairing up *all members* of A with *some members* of B (but not all).
– There is *no way* of pairing up all members of A with *all members* of B.

***Exercise 3*:** If A is a subset of B, then show that $A \leq B$.

Notice that proving $A = B$ is relatively easy because you just have to figure out *one possible way* to pair up all the members of the two sets.
Showing that $A < B$ is much trickier. It's not enough to show *some* attempt at pairing up the two sets doesn't work – *you need to show that every possible attempt at pairing will fail.*
This requires a good dose of ingenuity as we shall see later.

So, with these definitions in hand, we can go ahead and say that $\{1, 2, 3\} \leq \{a, b, c\}$ and $\{1, 2, 3\} \leq \{a, b, c, d, e\}$.
Furthermore, $\{1, 2, 3\} = \{a, b, c\}$, but $\{1, 2, 3\} < \{a, b, c, d, e\}$.

"Now this is all very interesting", you say, "but the fact is I do know how to count and I could have told you this much earlier. Why did we take such a long winded route to this easy conclusion?"

Because – and this was the great insight of George Cantor, father of Set Theory – this method of comparing sizes of sets by pairing works perfectly well for infinite sets, whereas counting fails miserably!!
In other words, we have just managed to extend the concept of counting from finite to infinite collections.[1]

So, without further ado, allow me to introduce …

[1] In fact, note that we were a bit vague when defining what an infinite set is. The technical definition is that a set S is infinite if and only if there is a bijection between S and a proper subset of S.

The Smallest Infinity

Let **N**= {0, 1, 2, 3, ...}, be the natural numbers. This is the smallest possible infinite set.
What do I mean by that exactly? Well, this.

***Theorem*:** For any infinite set A, $\mathbf{N} \leq A$.

***Proof*:** Pick any member from A, pair it up with 1. Now pick up another member, pair it up with 2. Keep going.
Now, if we run out of members to pick up at the k-th step for some number k, then A had only k members. But that means A was finite!

So, the process must continue until every member of **N** gets paired up with some members of A, but not necessarily all. Hence, by definition, $\mathbf{N} \leq A$.
QED

But are we really sure of this? After all remember the sets {2, 3, 4, 5 ...} and the odd numbers {1, 3, 5, 7,...}? We had suspected that these were *smaller* than **N**, but how is that possible if **N** is the smallest infinity?

The answer is that both the sets above have the same size as ***N****.*

Look at the pairing $0 \to 2$, $1 \to 3$, ..., $k \to k+2$,...
Convince yourself that using this formula every single member of {2, 3, 4, 5 ...} gets paired up with **N**. Hence they have the same size.

Similarly, $0 \to 1$, $1 \to 3$, $2 \to 5$, ..., $k \to 2k+1$, ...pairs up **N**and all the odd numbers.

So, here we have a property of infinite sets that has absolutely no analogue in the world of the merely finite:
It is possible to remove members from an infinite set – even infinitely many of them – and still end with a set of the same size!!

What about the other direction? What if we add more things to **N**?

How about S = {0, 0.5, 1, 1.5, 2, 3, 4, 5, ...} where we threw in two more numbers? Maybe that's a bigger infinity?

No such luck.

$0 \to 0$, $1 \to 1.5$, $2 \to 1$, $3 \to 1.5$, $4 \to 2$, $5 \to 3$, $k \to k-2$ for all subsequent numbers k, and we have paired up **N** and S with nothing left over.
In fact, you could have thrown in a million, a trillion or indeed, *any finite number* of extra members to **N** without changing its size at all!

Ok, then, let's get serious.
Let S = {0, 1/2, 1, 3/2, 2, 5/2, 3, 7/2, …} (every whole number and whole number + 1/2)
Now we have thrown in infinitely more things. Could this give something bigger?
Well, consider the mapping: $0 \to 0$, $1 \to 1/2$, $2 \to 1$, …, $k \to k/2$ for all numbers k and once again, our hopes are dashed as everything gets paired up.

It keeps getting worse.
You can add a million sets to **N**, or a gazillion – each having the same size as **N** – and still the size remains stubbornly unchanged!

In desperation, we go all the way.
Consider an infinite collection of sets, A_0, A_1, A_2 and so on, one for each whole number. Let each of these sets be individually as large as **N**.

How large is this entire infinite collection of sets taken altogether?
The completely stunning – but unequivocal – answer is:
Just as large as **N**, not one bit more!!

Of course, this needs a proof. So, here you go.
***Proof*:** Let the sets be given by: $A_0 = \{a_{00}, a_{01}, a_{02}, \ldots\}$
$A_1 = \{a_{10}, a_{11}, a_{12}, \ldots\}$ and similarly, for each n
$A_n = \{a_{n0}, a_{n1}, a_{n2}, \ldots\}$.
Visualize each set written out as an infinite row. All the sets written out this way looks like an infinite table which keeps going down forever and each row of the table stretches to the right forever.

Now here in the bijection from **N** to "Union of all A_n":
$0 \to a_{00}$, $1 \to a_{01}$, $2 \to a_{10}$, $3 \to a_{02}$, $4 \to a_{11}$, $5 \to a_{20}$, $6 \to a_{30}$, …
Do you see what we are doing?
We are traversing the table diagonally from upper right to lower left, and once

you see that, it will be clear that the map is a bijection. ***QED***

At this point, you may legitimately wonder why we called **N** the *smallest* infinity.
Every infinite set which looked like it might be smaller than **N** ended up being the same size.
On the other hand, every attempt to get something bigger than **N** failed as well.

Maybe it's time to call it quits and admit that there is only one infinity – the size of the set **N**.
Not quite.

A Bigger Infinity

Ladies and gentlemen, allow me to present:

***Cantor's Theorem*:** Let A be any set, finite or infinite. Then $A <$ Power set of A.

***Proof*:** Let's explicitly write down what we need to prove, first.

Let $P(A)$ denote the power set of A.
By the definition of "$A < P(A)$", we need to show two things
1) There is *some way* of pairing up *all members* of A with *some members* of $P(A)$ (but not all).
2) There is *no way* of pairing up all members of A with *all members* of $P(A)$.

Step 1) is easy.
Note that $P(A)$ consists of all possible subsets of A. In particular, for any member x belonging to A, the set $\{x\}$ belongs to $P(A)$. So, simply pair $x \to \{x\}$ for each x and we are done.
To illustrate using **N**, we just pair $0 \to \{0\}$, $1 \to \{1\}$, $2 \to \{2\}$ and so on...

Step 2) was Cantor's stroke of genius.
Remember, we need to show that *no conceivable pairing* between members of A and $P(A)$ will work.

So, let's say the World's Biggest Math Genius has given you a possible way of pairing them all up.
As per his formula:
– Every member x gets paired with some subset of A, which we will call $F(x)$.
– More importantly, for *any* subset S of A, there is some member x of A such that $S = F(x)$.

We will use Mr. Genius' pairing formula against him, as follows:
For any x, look at the subset $F(x)$ as per his formula.
Now ask the question, Does $F(x)$ contain x?
If the answer is "Yes", call x a "Good point".
If the answer is "No", call x a "Bad point".
Note that *every point x in A must be either Good or Bad, but not both.*

Then define: Rogue Subset = All Bad points in A.

To illustrate, suppose as per Mr. Genius's formula:
$0 \to \{1, 2, 3\}$, $1 \to \{1, 3, 5, 7, \ldots,$ all odd numbers$\}$ and $4 \to \{1, 6, 11, 13\}$.
Then, Rogue Subset would include 0 and 4, but not include 1.

Now take a moment to realize that:
The Rogue Subset cannot have been paired with any member of A.

How so? Let's see.
Suppose there is some x^* such that $F(x^*) =$ Rogue Subset.
Now x^* must be either a Good point or a Bad point.
If x^* is a Good point, then $F(x^*)$ must contain $x^* \implies$ Rogue Subset must contain x^*.
But Rogue Subset, by definition, consists only of Bad points $\implies x^*$is a Bad point.
Contradiction.

Ok, so then suppose x^* is a Bad point.
But then, by definition of Rogue Subset, x^* is contained in Rogue Subset.
But Rogue Subset $= F(x^*) \implies x^*$ is contained in $F(x^*) \implies x^*$ is a Good point.
Contradiction.

Only way to avoid a contradiction is to realize that *no point x^* in A was paired* *with the Rogue Subset.*

Note that this doesn't depend on any particular recipe for pairing.
If Mr. Genius realized his mistake and gave you another pairing recipe, then you would just follow the same procedure for *that* recipe and end up with *another* Rogue Subset.
Every possible attempt is doomed to failure and this completes Step 2)

Thus, we have shown that $A < P(A)$.
Done!! ***QED***

Let's just take a deep breath and appreciate the magnitude of what we just proved.
For finite sets, this result is not a big deal because $n < 2^n$ for any number n (Exercise 1).

But the true magic of Cantor's Theorem shines through when A is an infinite set.
For at last, we can break through the tyranny of $\mathbf{N}$ and soar upwards to a provably bigger infinity, the power set of $\mathbf{N}$.
With that small (or is it infinitely large?) step, our journey has begun.

The Ladder of Infinity

Start by renaming $\mathbf{N}$. Call it I_0 – the "zero-th infinity".
We now know that $\mathbf{N} < P(\mathbf{N})$, the power set of $\mathbf{N}$.
Call that I_1.

But what worked once, works again. So, $P(\mathbf{N}) < P(P(\mathbf{N}))$. Call that I_2.
Repeat. Repeat. Repeat again.

What you get is a chain of infinities, each bigger than the last – $I_0 < I_1 < I_2 < \cdots < I_{1000} < \ldots$
In other words, not only are there different sizes of infinity, you get *infinitely many different sizes – one for every whole number.*

(Quick Question: How do we know there are no infinities between I_0 and I_1?

Quick Answer: This is the Continuum Hypothesis and well beyond the scope of this article.)

Leaving already? But we are just getting started.

Remember how we are allowed to take unions of sets?
Well then, once we get our infinity of infinite sets, each bigger than the last, let's take the union them all!
Look at the Jumbo Set, $J = I_0 \cup I_1 \cup I_2 \cup \ldots$

J is a strictly bigger infinite set than all the I's!!

***Proof*:** Okay, suppose, instead, that $J = I_k$ for some integer k.
But that can't be, because J contains all the members of I_{k+1} and $I_k < I_{k+1}$.
So, $I_k < J$ for all I_k. ***QED***

So, we can think of J as the "infinity-th infinity", larger than anything that came before.
Let's call it I_ω.

But now the fun starts all over again – for nothing stops us from looking at $P(J)$, $P(P(J))$ and so on to get $I_\omega < I_{\omega+1} < I_{\omega+2} < \ldots$
And yes, when this chain gets exhausted, we take the union once again to get a super jumbo sized infinite set $I_{2\omega}$.

Power set, power set, power set, union, repeat, repeat...– $I_{3\omega}$, $I_{4\omega}$, $I_{5\omega}$,...
Union them all and out pops I_{ω^2}.
Plodding ever onwards, we get I_{ω^3}, I_{ω^4}, I_{ω^5} ...Then eventually I_{ω^ω}.
And beyond that as well...

At this point, you may turn around and ask:
"How many different sizes of infinity are there? How many steps on the ladder of infinity?"

Infinitely many steps, obviously, but by now we have learned to calibrate more finely.
I_0 different sizes of infinity? I_1 maybe? Surely not I_ω?

It can be proved that, in a very well defined sense, there is an I_1-th infinity, an

I_2-th infinity, an I_ω-th infinity, even an I_{ω^ω}-th infinity.

In fact, we can find an infinity – call it I_{Crazy} – such that I_{Crazy} is the I_{Crazy}-th infinity. In other words, $I_{Crazy} = I_{I_{Crazy}}$.
That's crazy! Yes, I know.

As for your question, the answer is that there are more rungs on the ladder of infinity than can be counted by *any* of the infinities lying on that ladder!

Let's say that once again – not only are there different sizes of infinity, there are far, far, far more of them than there are numbers.
In fact, there are more types of infinity than can be counted by any of the infinities in our ever increasing ladder.

Let us prove that completely mind-blowing fact I just mentioned.
Proof: Suppose there are I_{BIG} many different sizes of infinity.
In other words, there is an infinite set, I_{BIG}, such that every infinite set can be written as I_k for some element $k \in I_{BIG}$.
First, define $I^* = \cup_{k \in I_{BIG}} I_k$.
Now consider the set $P(I^*)$ = Power set of I^*.
It is easy to see that $P(I^*)$ is an infinite set and $P(I^*) >$ Every set I_k (Why?)
$\implies P(I^*)$ does not correspond to any I_k.
But, by assumption, the collection $\{I_k\}$ was supposed to contain every infinite set.
Contradiction. ***QED***

So, where does this leave us?

The Universe of Sets

Infinity has traditionally connoted something mighty and mysterious, possibly beyond human understanding.

But now that we are dealing with infinities in the plural, it becomes apparent that they behave rather like the familiar finite (whole) numbers.

Some are bigger than others.

Taking a power set always yields a bigger infinity, just like adding one always gives a bigger number.
The enormous profusion of infinities, with entities like I_{ω^ω} or I_{Crazy} may be startling, but perhaps no more so than the existence of really large numbers, like the googolplex or Graham's number.

It turns out we can even do arithmetic with these infinities – add them up, multiply them, take one infinity to the power of another and so on. The fact that there are more different sizes of infinity than can be counted by any given infinity is precisely analogous to there being more finite numbers than can be counted by any one number.

So can we now claim to have "tamed infinity", reducing the mysterious Infinite to simple arithmetic?
Well, let's just take one final step.
What if we take the entire collection of all possible sets – finite or infinite – and join them together to form the humongous "Set of all sets"? Okay, fine, call this "universal set" U.
Surely, by definition, nothing can be larger than U?
But wait, says Cantor, we just proved that for any set whatsoever, $P(A) > A$. So, $P(U)$ must be greater than U. Contradiction !

What just happened there? Just when we were smugly speaking of taming infinity, we seem to have tripped over our mathematical feet and fallen flat!

A detailed answer would take us too far afield into Axiomatic Set Theory, so let me just say this: It turns out that the collection of all sets is too vastly enormous to be conveniently encapsulated in curly brackets and called a "set". There is no "Set of All Sets".

What we have, instead, is the Universe of All Sets. Associated with the Universe is an "Infinity beyond all infinities" which Cantor referred to as the Absolute Infinite.
This is the Poetic Infinity that we spoke of at the start – the Ultimate Maximum that admits nothing greater!

A deeply religious man, Cantor associated the Absolute Infinite with God – in contrast to the garden variety infinities we have been discussing, which he

called the “transfinite cardinals”
or “transfinite numbers”.

So, at the end of our excursion into the set theoretic realm, we are left with:
– The finite numbers which we all know and love
– The transfinite numbers forming our “ladder of infinity”
– The Absolute Infinite encompassing and transcending them all.

The ladder of infinity climbs to dizzying heights and perched high on its transfinite rungs, the mathematician may be tempted to smile condescendingly at the puny finite numbers far below and the once-mighty infinite set **N**.
Until, struck by a sudden thought, he looks upwards to behold the starry firmament of the Absolute, far above and forever inaccessible...

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8 The Cartan-Dieudonné Theorem about Orthogonal Matrices *By J. K. Verma*

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8.1 Introduction

The purpose of this note is to explain a beautiful theorem of Elié Cartan about decomposition of orthogonal matrices as product of matrices of reflections. This was generalized to matrices with entries in an arbitrary field by J. Dieudonné. Hence it is called the Cartan-Dieudonné Theorem.

In order to state the theorem, we first fix notation and recall a few definitions. Let $\mathbb{R}$ denote the field of real numbers. The dot product of two column vectors $u, v \in \mathbb{R}^n$ is defined as $u.v = u^t v$ where A^t denotes the transpose of any matrix A. A basis $u_1, u_2, \ldots, u_n$ of $\mathbb{R}^n$ is called an *orthonormal basis* of $\mathbb{R}^n$ if $u_i.u_j = 0$ for $i \neq j$ and $u_i.u_i = 1$ for all $i = 1, 2, \ldots, n$. An $n \times n$ real matrix A is said to be *orthogonal* if the column vectors of A form an orthonormal basis of $\mathbb{R}^n$. Equivalently A is orthogonal if $A^t A = I$. It is easy to show that the row vectors as well as the column vectors of an orthogonal matrix constitute an orthonormal basis of $\mathbb{R}^n$.

Example 8.1. Let A be a 2×2 orthogonal matrix. Since the column vectors are mutually perpendicular unit vectors in the plane, it follows that

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ or } A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

for some $\theta \in [0, 2\pi]$. The first matrix represents a linear transformation of $\mathbb{R}^2$ which maps any vector $w \in \mathbb{R}^2$ to a vector obtained by rotating w by an angle θ in anti-clockwise direction. The second matrix represents a linear transformation of $\mathbb{R}^2$ which reflects vectors with respect to the line $y = x\tan(\theta/2)$.

8.1.1 Reflections in $\mathbf{R}^n$.

We construct matrices which represent reflection in $\mathbb{R}^n$ with respect to a hyperplane. Recall that a hyperplane is a subspace of $\mathbb{R}^n$ of dimension $n-1$. Thus a hyperplane in $\mathbb{R}^2$ is a line passing through the origin and a hyperplane in $\mathbb{R}^3$ is a plane passing through the orgin. Let u be a unit vector in $\mathbb{R}^n$. Let I denote

[1]These are notes of two lectures delivered in the *Training Programme in Mathematics* during 3-4 June, 2018 at NISER, Bhubaneswar

the $n \times n$ identity matrix. Consider the *Householder* matrix of u,

$$H = I - 2uu^t.$$

Then $Hu = u - 2u(u^t u) = -u$. If $w \perp u$ then $Hw = w - 2uu^t w = w$. Let $L(u)$ denote the linear span of the vector u. For any subspace V of $\mathbb{R}^n$, define:

$$V^\perp = \{u \in \mathbb{R}^n \mid u.v = 0 \quad \forall\, v \in V\}$$

Proposition 8.2. *For any subspace V of $\mathbb{R}^n$, we have $\mathbb{R}^n = V \oplus V^\perp$.*

Proof. It is clear that $V \oplus V^\perp \subset \mathbb{R}^n$. Let $\{v_1, v_2, \ldots, v_m\}$ be an orthonormal basis of V. Define $T : \mathbb{R}^n \longrightarrow V$ by:

$$Tu = \sum_{i=1}^{m} (u.v_i)v_i$$

The set $\{v_1, \ldots, v_m\}$ can be extended to an orthonormal basis $\{v_1, \ldots, v_m, \ldots, v_n\}$ for $\mathbb{R}^n$. T is clearly onto and

$$\ker(T) = \{u \in \mathbb{R}^n \mid u.v_i = 0 \quad \forall\, i = 1, 2, \ldots, m\} = V^\perp.$$

Hence by the rank-nullity theorem, $\dim V + \dim V^\perp. = n$ Note that $u = Tu + (u - Tu)$. We show that $u - Tu \in V^\perp$.

$$\begin{aligned}(u - Tu).Tu &= u.Tu - Tu.Tu \\ &= \left(\sum_{i=1}^{n} (u.v_i)v_i\right) \cdot \left(\sum_{i=1}^{m} (u.v_i)v_i\right) - \sum_{i=1}^{m} (u.v_i)^2 \\ &= 0\end{aligned}$$

Hence $(u - Tu) \perp Tu$. Hence $\mathbb{R}^n \subset V \oplus V^\perp$. Therefore $\mathbb{R}^n = V \oplus V^\perp$. □

By this proposition $\mathbb{R}^n = L(u) \oplus L(u)^\perp$. Hence $L(u)^\perp$ is an $(n-1)$- dimensional subspace of $\mathbb{R}^n$ which is perpendicular to $L(u)$.

Definition 8.3. *A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a reflection with respect to a hyperplane H of $Tu = -u$ where $u \perp H$ and $Tv = v$ for all $v \in H$.*

Thus $H = I - 2uu^t$ is a reflection with respect to the hyperplane $L(u)^\perp$. Moreover H is an orthogonal symmetric matrix with $H^2 = I$. Indeed,

$$\begin{aligned}H^t H &= \left(I - 2uu^t\right)\left(I - 2uu^t\right) \\ &= I - 2uu^t - 2uu^t + 4uu^t uu^t \\ &= I.\end{aligned}$$

We shall denote length of a vector u by $||u||$.

Definition 8.4 (Orthogonal Transformation). *A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called orthogonal if $\|Tu\| = \|u\|$ for all $u \in \mathbb{R}^n$.*

Theorem 8.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then the following are equivalent:*
(1) T is an orthogonal transformation.
(2) $Tu.Tv = u.v$ for all $u, v \in \mathbb{R}^n$.
(3) If $\{u_1, u_2, \ldots, u_n\}$ is an orthonormal basis of $\mathbb{R}^n$ then $\{Tu_1, Tu_2, \ldots, Tu_n\}$ is an orthonormal basis of $\mathbb{R}^n$.
(4) The matrix A of T with respect to any orthonormal basis of $\mathbb{R}^n$ is orthogonal.

Proof. (1)⇒(2). Let T be an orthogonal transformation. Since $\|T(u+v)\| = \|u+v\|$ for all $u, v \in V$, we have

$$\begin{aligned}(Tu+Tv).(Tu+Tv) &= Tu.Tu + 2Tu.Tv + Tv.Tv \\ &= u.u + 2u.v + v.v\end{aligned}$$

Hence $Tu.Tv = u.v$.
(2)⇒(3). Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then $\{Tu_1, \ldots, Tu_n\}$ is also an orthonormal basis of $\mathbb{R}^n$ since T preserves the dot product of vectors.
(3)⇒(1). Suppose for some orthonormal basis $\{u_1, \ldots, u_n\}$, $\{Tu_1, \ldots, Tu_n\}$ is also an orthonormal basis. Let $u = x_1u_1 + \cdots + x_nu_n$ for some $x_1, \ldots, x_n \in \mathbb{R}$. Then

$$\begin{aligned}Tu &= x_1Tu_1 + \cdots + x_nTu_n \\ Tu.Tu &= x_1^2 + \cdots + x_n^2 = \|u\|^2\end{aligned}$$

(3) ⇒ (4). Let $B = \{u_1, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{R}^n$ and $Tu_j = \sum_{i=1}^n a_{ij}u_i$ for $j = 1, 2, \ldots, n$. We show that $A = (a_{ij})$ is an orthogonal matrix. Since $Tu_1, \ldots, Tu_n$ is an orthonormal basis, we get

$$1 = Tu_j.Tu_j = \left(\sum_{i=1}^n a_{ij}u_i, \sum_{i=1}^n a_{ij}u_i\right) = \sum a_{ij}^2.$$

For $j \neq k$, we have

$$Tu_j.Tu_k = \left(\sum_{i=1}^n a_{ij}u_i, \sum_{i=1}^n a_{ik}u_i\right) = \sum_{i=1}^n a_{ij}a_{ik} = 0.$$

(4)⇒(3). If A is orthogonal then the above equations show that T maps an orthonormal basis to an orthonormal basis.

□

8.2 The Cartan-Dieudonné Theorem

We prove the Cartan-Dieudonné Theorem for finite dimensional real inner product spaces. Readers who are not familiar with this notion may replace these spaces with $\mathbb{R}^n$ with the dot product of vectors.

Theorem 8.6 (**Cartan-Dieudonné Theorem**). *Every orthogonal transformation of a real inner product space of dimension n is a product of at most n reflections.*

Proof. Apply induction on $n = \dim V$. Suppose $n = 1$. Then $V = L(u)$ for some unit vector u. Therefore $\|Tu\| = 1$. Hence $Tu = \pm u$. Now let $n \geq 2$ and assume the theorem holds true for orthogonal transformations on $(n-1)$-dimensional real inner product spaces.

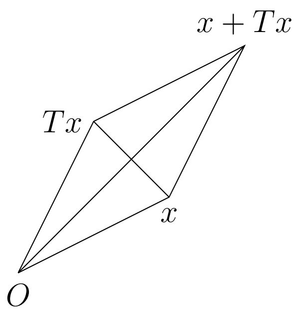
Case 1: Suppose $Tx = x$ for a non-zero $x \in V$ and $u \in L(x)^{\perp} = W$. Then

$$(Tu, x) = (Tu, Tx) = (u, x) = 0.$$

Hence $Tu \in L(x)^{\perp}$. Hence $T : W \to W$ is an orthogonal transformation of an $(n-1)$-dimensional inner product space W. By the induction hypothesis, there exist reflections of W say $s_1, s_2, \ldots, s_r$ such that $r \leq n-1$ and

$$T|_W = s_1 s_2 \cdots s_r.$$

Define $T_i : V \to V$ by $T_i(x) = x$ and $T_i|_W = s_i$. Let H_i be the hyperplane in W fixed by s_i. Then $V_i = H_i \oplus L(x)$ is fixed by T_i. Let $H_i^{\perp} \subset W$ and $H_i^{\perp} = L(x_i)$ for $i = 1, 2, \ldots, r$. Note that $x_i \perp x$ and $x_i \perp H_i$. Hence $x_i \in V_i^{\perp}$. Thus $V = V_i \oplus L(x_i)$. We show that each T_i is a reflection in V. Since $T_i(x_i) = s_i(x_i) = -x_i$ and $T_i(u) = u$ for all $u \in V_i$, therefore T_i is a reflection. From the definition of T_i, it follows that $T = T_1 T_2 \cdots T_r$.



Case 2 : Suppose $Tx \neq x$ for any $x \in V$. Then $u = Tx - x \neq 0$ and let $H = L(u)^{\perp}$ and U be a reflection with respect H. Then

$$\begin{aligned}(Tx + x, Tx - x) &= (Tx, Tx) + (x, Tx) - (Tx, x) \\ &\quad - (x, x) \\ &= 0\end{aligned}$$

So $Tx + x \perp u \Rightarrow Tx + x \in H$. Hence $U(Tx + x) = Tx + x$ and $U(Tx - x) = x - Tx$. Note that

$$\begin{aligned} U(Tx + x) &= UTx + Ux &= Tx + x \quad \text{and} \\ U(Tx - x) &= UTx - Ux &= x - Tx \end{aligned}$$

Therefore $UTx = x$ and by Case 1, $UT = T_1T_2\cdots T_s$ for some reflections $T_1, T_2, \ldots, T_s$ of V and $s \leq n-1$. Hence

$$UUT = T = UT_1T_2\cdots T_s$$

is a product of at most n reflections. □

Corollary 8.7. *Let A be a 3×3 orthogonal matrix with* $\det A = 1$. *Then* 1 *is an eigenvalue of A.*

Proof. If $A = I$ then the result is clear. So let $A \neq I$. Let T be the orthogonal transformation of $\mathbb{R}^3$ induced by A. Since $\det A = 1$, by the Cartan-Dieudonné Theorem, $T = R_1R_2$ where R_1 and R_2 are distinct reflections with respect to planes H_1 and H_2 respectively. Then

$$\dim(H_1 + H_2) = 3 = \dim H_1 + \dim H_2 - \dim(H_1 \cap H_2) = 4 - \dim(H_1 \cap H_2).$$

It follows that $\dim(H_1 \cap H_2) = 1$. Since the vectors in H_1 and H_2 are fixed by R_1 and R_2 respectively, any $u \in H_1 \cap H_2$ is fixed by T. Thus $Tu = u$. Hence u is an eigenvector of A with eigenvalue 1. □

Remark 8.8. *The orthogonal matrices of size $n \times n$ form a group under multiplication. It is called the orthogonal group and it is denoted by $O(n)$. The subgroup of $O(n)$ consisting of matrices of determinant* 1 *is called the special orthogonal group. It is denoted by $SO(n)$. Euler proved that the rotations of* $\mathbb{R}^3$ *about an axis passing through the origin are given by the linear transformations of* $\mathbb{R}^3$ *induced by the matrices in $SO(3)$. This is known as* Euler's Theorem *in Mechanics. This fact can easily be deduced from the corollary proved above.*

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9 Solutions to problems in Blackboard issue I

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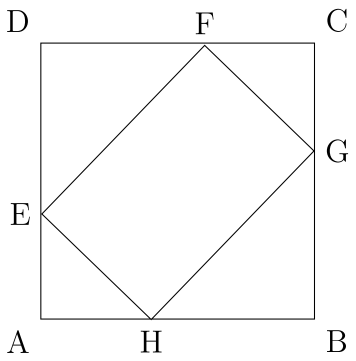
Q 1. Solve ACID + BASE = SALT + H2O where each letter stands for a distinct digit.

We leave this simple problem to the reader.

Q 2. An $a \times b$ rectangular painting needs to be covered by a square frame. Find the side length of the smallest square frame required.

Solution.
Without loss of generality, let $a \leq b$. We claim that the smallest square frame has side length $(a+b)/\sqrt{2}$ or b according as to which is smaller.



Let $ABCD$ be the smallest square frame which covers the rectangular painting $EFGH$ completely. We may assume that three of the vertices, say E, G, H of the painting rest on the sides of the square frame. If θ is the acute angle $\angle EHA$, and d is the distance of F from the side CD, then the sides $AB = BC$ equal

$$a\cos\theta + b\sin\theta = d + a\sin\theta + b\cos\theta$$

We need to minimize $a\cos\theta + b\sin\theta$ as a function of θ. In fact, since $d \geq 0$, this means we have $\sin\theta \geq \cos\theta$ which gives $\pi/4 \leq \theta$. Putting the derivative $-a\sin\theta + b\cos\theta = 0$ in the range $[\pi/4, \pi/2]$, we have local extrema $\pi/4, tan^{-1}(b/a)$ and $\pi/2$. These give the values $(a+b)/\sqrt{2}$, $\sqrt{a^2+b^2}$ and b respectively. Clearly, the middle value is at least as large as b which gives our minimum as the smaller number among $(a+b)/\sqrt{2}$ and b.

Q 3. Find the smallest positive integer N so that the sum $\sum_{n=1}^{N} \frac{1}{n!+(n+1)!} >$ 0.49995.

Solution.
As

$$\frac{1}{n!+(n+1)!}=\frac{1}{n!(n+2)}=\frac{n+1}{(n+2)!}=\frac{(n+2)-1}{(n+2)!}=\frac{1}{(n+1)!}-\frac{1}{(n+2)!},$$

the sum is telescoping. The sum from $n=1$ to $n=N$ then equals $\frac{1}{2}-\frac{1}{(N+2)!}$. If this is bigger than 0.49995, we easily obtain $(N+2)!>10^5/5$ and hence $N\geq 6$. The smallest value of N is 6.

Q 4. In the multiplication table below, each digit from 0 to 9 appears exactly twice. Determine all such tables.

$$\begin{array}{rrrrr}
 & & * & * & * \\
 & \times & * & * & * \\
\hline
 & & * & * & * \\
 & * & * & * & \\
* & * & * & & \\
\hline
* & * & * & * & * \\
\hline
\end{array}$$

This problem is also left to the reader; there is a unique answer.

Q 5. Determine the sum of the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\frac{1}{1807}+\cdots$

Solution.
Our sequence is $a_1=2$ and $a_{n+1}=\prod_{i=1}^{n}a_i+1$. Then one has $\sum_{i=1}^{n}\frac{1}{a_i}+\prod_{i=1}^{n}\frac{1}{a_i}=1$ for each n, by induction on n. From this, it is clear that the infinite series converges to 1.

Q 6.
(a) Prove the identity $\sum_{n\geq 1}\frac{1}{n^n}=\int_0^1\frac{dx}{x^x}$.
(b) Prove the identity $\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^n}=\int_0^1 x^x dx$.

Solution.
$\frac{1}{x^x}=e^{-x\log(x)}=\sum_{n\geq 0}\frac{(-1)^n(x\log(x))^n}{n!}$ where the series converges uniformly on $[0,1]$. Integrating from 0 to 1 and interchanging integral and sum we obtain $\int_0^1\frac{dx}{x^x}=\sum_{n\geq 0}\frac{(-1)^n}{n!}\int_0^1(x\log(x))^n dx$.
Now, more generally, $I_{m,n}=\int_0^1 x^n\log(x)^m dx$ satisfies the recursion $I_{m,n}=\frac{-m}{n+1}I_{m-1,n}$. From this, we finally obtain $I_{m,n}=\frac{(-1)^m m!}{(n+1)^{m+1}}$. Our integral is $I_{n,n}=\frac{(-1)^n n!}{(n+1)^{n+1}}$ which gives

$$\int_0^1\frac{dx}{x^x}=\sum_{n\geq 0}\frac{1}{(n+1)^{n+1}}.$$

The identity for $\frac{(-1)^n}{n^n}$ follows similarly.

Q 7. Let f be a function from $\mathbb{R}$ to itself. Suppose that for each $a \in \mathbb{R}$, the number of elements in $\{t \in \mathbb{R} : f(t) = a\}$ is 0 or 2. Prove that f has infinitely many points of discontinuity.

Solution.
Assume that there are only finitely many points of discontinuity. If f is continuous on an open interval, then it must be monotone in a neighbourhood of all except two points in it. Indeed, if f changed direction at points $a < b < c$; say, $f(a) < f(c) < f(b)$, then for a small enough $t > 0$, the value $f(c) + t$ is a value taken at at least 3 points by the intermediate value theorem. Hence, the set S of all points where either f is discontinuous or changes direction in a neighbourhood, is finite. Hence the set $T = f^{-1}(f(S))$ is also finite. The complement of T is a union of $|T| + 1$ open intervals and the complement of $f(T)$ is a union of $|f(T)| + 1 = |T|/2 + 1$ disjoint open intervals; call these sets U and V. Then, the function f from U to V is monotonoe and two-to-one. By connectedness, each of the $|T| + 1$ intervals maps completely into a single inrerval among the $|T|/2+1$ intervals that make up V. Since $2(|T|/2+1) > |T|+1$, by the pigeon-hole principle, there is an interval I in V which has exactly one corresponding interval in U containing the preimages of I. This contradicts the hypothesis that every preimage has cardinality 0 or 2. Thus, there are infinitely many points of discontinuity.

Q 8. A deck of 52 cards is given. There are four suites each having cards numbered $1, 2, \cdots, 13$. The audience chooses any five cards with distinct numbers written on them. The assistant of the magician comes by, looks at the five cards and turns exactly one of them face down and arranges all five cards in some order. Then the magician enters and with an agreement made beforehand with the assistant, she has to determine the face down card (both suite and number). Explain how the trick can be completed.

Solution.
The assistant performs the following trick. By the Pigeonhole principle, some two of the five cards must have the same suite. She turns one of them face down and arranges the face down and face up card in a way so that the smaller card is to the left. Next she takes the remaining three cards and puts them to the right of these two and arranges them. Since they have three distinct numbers on them, there are 3! ways to arrange them. In the agreement made beforehand, each of these arrangements corresponds to a number between 1 and 6 since only the order of the cards matter. Now, the magician enters. She sees that the face down card is either to the left most or second from the left. In both cases she

knows the card has the same suite as the face down card. Thus, she can identify the suite of the face down card. By the placement he can identify whether the face down card has a larger or a smaller number than the other card of its suite. Now, looking at the third, fourth and fifth card from the left, and the agreement on numbers, she can get a number between 1 and 6. The assistant arranges them so that this number is precisely the distance between the face down card and the card of the same suite when all the thirteen cards are put on a circle. Since the magician also knows which card has a larger number and the difference between them on a circle, she can find the face down card.

Q 9. Find all polynomials p which satisfy the property that a value $p(a)$ is rational if and only if a is rational.

Solution.
Write $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_n \neq 0$. Putting the (rational) values at the points $0, 1, \cdots, n$ say, one can solve for the a_i's in rational numbers. We assume without loss of generality that $a_n > 0$. We claim that $n = 1$. By the theorem on rational roots, it follows that any a such that $p(a)$ is an integer must be of the form r/a_n. Hence, any two points a, b such that $p(a), p(b)$ are integers must be at least $1/a_n$ apart. Now, if $n \geq 2$, then $p'(x)$ tends to infinity as $n \to \infty$. Hence, there exists a such that $p(a + 1/a_n) - p(a)$ is large, say > 2. But then there are at least two distinct points within $1/a_n$ apart taking different integer values. This contradicts the observation made above.

Q 10. Let G be a finite group of order n. For any subset S of G, put

$$S^k = \{s_1 \cdots s_k : s_i \in S\}$$

for each $k \geq 1$. Prove that S^n is always a subgroup.

Solution.
For any $g \in S$ and any $k > 0$, clearly $gS^k \subset S^{k+1}$ which gives

$$|S| \leq |S^2| \leq |S^3| \leq \cdots\cdots$$

Since G is finite, there is some k so that $|S^k| = |S^{k+1}|$. But then we also have

$$|S^{k+2}| = |S^{k+1}S| = |gS^k S| = |gS^{k+1}| = |S^{k+1}|.$$

Hence, $|S^k| = |S^l|$ for all $l > k$. As $O(G) = n$, cardinality of S^r is $\leq n$ for any r, which implies that $|S^n| = |S^m|$ for all $m > n$. In particular, $|S^n| = |S^{2n}|$ which gives on using the fact that $e \in S^n$, that $S^n \subset S^{2n}$. Thus, $S^n = S^{2n}$ which shows that S^n is closed under the group operation. The other properties required for a subgroup are evidently true.

Q 11. Let us mark off points on the unit circle, dividing the circumference into n equal parts where $n > 2$. Fix one of these points and, moving clockwise along the circumference, join this point to the k-th point for each k coprime to n. What is the products of the lengths of these chords?

Solution.
We may assume that the origin is the centre and that points are the n-th roots of unity $P_{d+1} = e^{2id\pi/n}$ for $d = 0, 1, \cdots, n-1$. Note that the product of lengths of all the chords P_1P_i is simply $\prod_{d=1}^{n-1} |1 - e^{2id\pi/n}|$. Since the polynomial $1 + X + \cdots + X^{n-1}$ has as roots all the n-th roots of 1 excepting 1 itself, we have

$$\prod_{d=1}^{n-1}(1 - e^{2id\pi/n}) = n$$

by evaluating at $X = 1$. Notice that we have the equality $\prod_{d=1}^{n-1}(1 - e^{2id\pi/n}) = n$ as complex numbers; that is, even without considering absolute values.
Now, let us consider our problem. Here, the product considered is

$$\prod_{(d,n)=1} |1 - e^{2id\pi/n}|.$$

Writing $P(n) = \prod_{l=1}^{n-1}(1 - \zeta^l) = n$ and $Q(n) = \prod_{(d,n)=1}(1 - \zeta^d)$, where $\zeta = e^{2i\pi/n}$, we can see that

$$P(n) = \prod_{r|n} Q(r).$$

By Möbius inversion, $Q(n) = \prod_{d|n} P(d)^{\mu(n/d)} = \prod_{d|n} d^{\mu(n/d)}$. The function

$$\log Q(n) = \sum_{d|n} \mu(n/d) \log(d)$$

can be identified with the von Mangoldt function $\Lambda(n)$ which is defined to have the value $\log(p)$ if n is a power of p and 0 otherwise. To see why $\Lambda(n) = \sum_{d|n} \mu(n/d) \log(d)$, we write $n = \prod_{p|n} p^{v_p(n)}$ and note that

$$\log(n) = \sum_{p|n} v_p(n) \log(p)$$

But, the right hand side is clearly $\sum_{d|n} \Lambda(d)$. Hence, Möbius inversion yields

$$\Lambda(n) = \sum_{d|n} \log(d)\mu(n/d).$$

Using this identification, exponentiation gives also the value asserted in the proposition; viz., $Q(n) = p$ or 1 according as to whether n is a power of a prime p or not.

Q 12. Wieferich observed in 1909 that if p is a prime for which Fermat's equation $x^p + y^p = z^p$ has a solution in positive integers which are not multiples of p, then $2^{p-1} \equiv 1 \bmod p^2$. Such primes are called Wieferich primes. Prove that a prime which is Mersenne or Fermat cannot be a Wieferich prime. Here Mersenne primes are those of the form $2^n - 1$ and Fermat primes are those of the form $2^m + 1$.

Solution.
We prove firstly:
Lemma. *Let p be a prime whose expression in a base $b > 1$ is of the form*

$$1 + b^k + b^{2k} + \cdots + b^{nk}$$

for some $n, k \geq 1$. Then, $b^{p-1} \equiv 1 + \frac{p-1}{(n+1)k}(b^k - 1)p \not\equiv 1$ mod p^2.
Proof.
Now $p = 1 + b^k + \cdots + b^{nk} = \frac{b^{(n+1)k}-1}{b^k-1}$.
Now, p and $b^k - 1$ are relatively prime because p is a prime and $p \geq b^k + 1 > b^k - 1$.
Since p divides $b^{(n+1)k} - 1$, the order of b mod p is a divisor of $(n+1)k$. If it were smaller, say mr, with $m|(n+1)$ and $r|k$, then either $m < n+1$ or $r < k$.
If $r < k$, then the assertion $b^{(n+1)r} \equiv 1$ mod p means p divides $(1 + b^r + \cdots + b^{nr})(b^r - 1)$.
Now, p and $b^r - 1$ are relatively prime because p is a prime and $p \geq b^k + 1 > b^r - 1$.
Hence $p = 1 + b^k + \cdots + b^{nk}$ divides $1 + b^r + \cdots + b^{nr}$, which is impossible as p is the bigger number.
Now, if $m < n+1$, then the condition $b^{mk} \equiv 1$ means p divides $1 + b^k + \cdots + b^{(m-1)k} = \frac{b^{(mk}-1}{b^k-1}$ as p and $b^k - 1$ are relatively prime because p is a prime and $p \geq b^k + 1 > b^k - 1$.
This is impossible, as $p = 1 + b^k + \cdots + b^{nk}$ is larger than $1 + b^k + \cdots + b^{(m-1)k}$.
We have shown that the order of b mod p is $(n+1)k$; hence, this order $(n+1)k$ divides $p - 1$.
Now, raise $b^{(n+1)k} = 1 + p(b^k - 1)$ to the $\frac{p-1}{(n+1)k}$-th power. We have

$$b^{p-1} \equiv 1 + p(b^k - 1)\frac{p-1}{(n+1)k} \quad mod \quad p^2$$

Now, again the observation that p is relatively prime to $b^k - 1$ implies that p does not divide $(b^k - 1)\frac{p-1}{(n+1)k}$.
This completes the proof.

As a corollary, we obtain:
A prime p of the form $b^N + 1$ or of the form $1 + b + b^2 + \cdots + b^n$ cannot satisfy $b^{p-1} \not\equiv 1$ mod p^2. In particular, neither Mersenne primes not Fermat primes

can be Wieferich primes.

Proof. Both cases considered are of the form mentioned in the lemma with $b = 2$.

Q 13. Prove that there is no function $f : \mathbf{R} \to \mathbf{R}$ which is continuous at every rational number and discontinuous at every irrational number.
(The question was phrased incorrectly earlier.)

Solution.
This follows from the Baire category theorem which implies that the set of real numbers is not meagre (that is, a countable union of nowhere dense sets). Indeed, that implies that the set of irrationals is meagre (else, $\mathbf{R} = \mathbf{Q} \cup (\mathbf{R} \setminus \mathbf{Q})$ would be meagre, a contradiction). Using Baire category theorem again, we can show that the set of points of discontinuity (of any function between metric spaces) is either meagre or has non-empty interior. The brief reasoning is as follows:
For any set S, the oscillation of f on S (denoted $\omega_f(S)$) is the diameter of $f(S)$. Then, the oscillation $\omega_f(x)$ of f at a point x is defined as the infimum of the oscillations $\omega_f(B(x,\epsilon))$ as $\epsilon \to 0$ - it could be infinite. Hence, f is continuous at x if and only if $\omega_f(x) = 0$. In general, this oscillation gives a measure of how discontinuous f is at x. In fact, if $t > 0$ and $\omega_f(x) < t$, then $\omega_f(y) < t$ for y in an open neighbourhood of x. Thus, the sets $\{x : \omega_f(x) < t\}$ are open as t varies. Varying t over $1/n$, we find that the set of points of discontinuity of f is the intersection of the countably many open sets $\{x : \omega_f(x) < 1/n\}$ over n. This proves that the complement of such a set is a countable union of closed sets. If at least one of these sets has a non-empty interior, so does the set of points of discontinuity. Otherwise, it is a meagre set. As the set of irrationals is not meagre (as observed above) and has empty interior, it follows that it cannot be the set of points of discontinuity of a function.

Q 14. Find the significance of the number

$$4959866698915122609810424451 2918.$$

Solution.
A. Cohn observed that if $a_n \cdots a_1 a_0$ are the base 10 digits of a prime number, then the polynomial $f(x) = a_0 + a_1 X + \cdots + a_n X^n$ is irreducible. This is a nice exercise but one may ask what happens if we allow the coefficients a_i's of $f(x)$ to be larger than 9 with $f(10)$ prime; is the result still valid? Michael Filaseta and others generalized Cohn's result and proved the remarkable result that there exists an integer polynomial f of degree 129 (explicitly written down)

whose largest coefficient is

$$49598666989151226098104244512919$$

such that $f(10)$ is prime but f is divisible by $x^2-20x+101$. Furthermore, every integer polynomial (of whatever degree) whose coefficients are non-negative and bounded by the above number minus 1 must be irreducible if it takes a prime value at 10. Readers interested in details of proof of this amazing result may consult a paper in the Journal of Number Theory titled aptly as 49598666989151226098104244512919.

Q 15. Determine whether the series $\sum_n \frac{1}{n^3 \sin^2(n)}$ converges or not.

Solution.
The answer to this question is still unknown! It depends on what is known as the irrationality measure of π. For any positive real, irrational number x, consider the infimum $\mu(x)$ of all possible positive integers n for which $|x - p/q| < 1/q^n$ can hold only for finitely many coprime positive integers p, q. For any real $x > 0$, this number $\mu(x)$ is called its irrationality measure and equals 1 only for rational x. If x is irrational, then $\mu(x) \geq 2$ (Roth was awarded the Fields medal for showing that it is 2 when x is algebraic), One does not know $\mu(\pi)$ as yet. Since $|x| \leq |\sin(x)| \leq \frac{2}{\pi}|x|$ when $|x| \leq \pi/2$, one can show that for positive real numbers u, v one has

$$n^u |\sin(n)|^v = O(1/n^{u-(\mu(\pi)-1)v-\epsilon})$$

for all $\epsilon > 0$, where the big-Oh notation means that the LHS is bounded by a constant times the RHS where the constant does not depend on n. From this, it follows that the sequence $\frac{1}{n^u|\sin(n)^v|}$ converges to 0 if $\mu(\pi) < 1+u/v$ and diverges if $\mu(\pi) > 1 + u/v$. Therefore, if the series $\sum_n \frac{1}{n^u|\sin(n)^v|}$ were to converge, we must necessarily have $\mu(\pi) \leq 1 + u/v$. We do not know the converse though. In our case, this would mean $\mu(\pi) \leq 5/2$ for our series to converge. This is unknown as yet though we expect $\mu(\pi) = 2$.

10 A Brief Introduction to the Mathematics of the *Śulbasūtras* *By Aditya Kolachana, K. Mahesh, and K. Ramasubramanian*

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Abstract

The *Śulbasūtras* are the oldest surviving mathematical texts in India that describe various geometrical procedures involved in the construction of altars used for performing a variety of vedic sacrifices. Herein, we find methods to construct various geometrical figures, the statement of the so-called Pythagorean theorem, techniques to transform a circle into a square and vice-versa, various methods to combine and transform areas, statements of areas and volumes for various figures, approximate values for surds, and solutions to indeterminate problems, among other topics. In this paper, we give a brief introduction to these important category of texts and describe a few of the geometrical techniques discussed in them. In a subsequent paper, we plan to take up other interesting topics such as the treatment of surds, construction of *citis* and so on that are dealt with in the *Śulbasūtras*.

10.1 Introduction

While the exact origins of mathematics in India are lost in the mists of time, the Vedas—among the oldest literature of mankind—give us some idea of the use of mathematics in those early times. The vedic people worshipped nature and propitiated it in its multifarious forms through the performance of different types of sacrifices, usually performed on altars referred to as *vedis* or *citis*. The *citis* constructed by the priests widely varied from one another in terms of size as well as shape.[1] Moreover, the vedic texts also imposed various constraints regarding the areas of different altars with respect to one another, the size and number of bricks involved in the construction, etc. Thus, the need for constructing complicated altars probably spurred the study of geometry in India in the early vedic period. The priests developed manuals to assist them in the construction of such altars, and this gave rise to a class of texts called the *Śulbasūtras*,[2] some of which are estimated to have been composed before 800 BCE.

[1]Some of the geometrical constructions such as the *śyenaciti*—an altar in the shape of a falcon in its flight—prescribed by the *Śulbakāras* (the authors of the *Śulbasūtras*) are beautiful structures and are non-trivial in their construction.

[2]The word *śulba* stems from the root *śulb* which means 'to measure'. Earlier, all measurements were done using ropes or cords and in due course the word *śulba* was employed to refer

The *Śulbasūtras* are considered to be a part of a larger class of texts known as *Kalpasūtras*, which in turn are considered to be one of the six *Vedāṅgas*.[3] The *Kalpasūtras*, more often simply referred to as *Kalpa*, also include *Gṛhya*, *Dharma* and *Śrautasūtras*, besides the *Śulbasūtras*. While *Dharmasūtras* deal with codes of conduct and general moral precepts, the *Gṛhya* and the *Śrautasūtras* lay down specific rules connected with the actual performance of rituals. The *Śulbasūtras* present the necessary mathematical tools and practical guidelines that would be required in the construction of altars employed in the performance of rituals.

The present paper is intended to provide a glimpse of the geometry involved in the construction of altars, and also a flavour of how the vedic people used to codify rules in the form of short aphorisms known as *sūtras*.[4] Here, we first present a brief note on the currently extant *Śulbasūtras* and their contents in Section 10.2. In Section 10.3, we discuss the description of the so-called Pythagorean theorem in the *Śulbasūtras* and investigate whether the priests were aware of the proof of this theorem. In Sections 10.4 and 10.5, we show how the *Śulbasūtras* tackled problems involving combining and transforming areas, and conclude this paper with our remarks in Section 10.6.

10.2 The extant *Śulbasūtras*

Scholarly investigations have shown that seven *Śulbasūtras*, namely *Bodhāyana*, *Āpastamba*, *Kātyāyana*, *Maitrāyaṇa*, *Mānava*, *Vārāha*, and *Vādhula* are extant today. Of these, the *Bodhāyana*, *Āpastamba*, and *Kātyāyana Śulbasūtras* have been studied more extensively by scholars.

The *Bodhāyana-śulbasūtra*[5] is perhaps the oldest[6] and also the largest, consisting of over 500 *sūtras* divided into three chapters. Some of the topics covered here can be seen in Table 1. It has two important commentaries, one by Vyaṅkateśadīkṣita called *Bodhāyanaśulba-mīmāṃsā*, and the other by

to the cords themselves. The term *sūtra* signifies the aphoristic style of composition that has been adopted by the authors in composing their texts.

[3]The term *Vedāṅga* is used to refer to six branches of knowledge namely *śikṣā*, *vyākaraṇam*, *kalpaḥ*, *niruktaṃ*, *jyotiṣaṃ* and *chandaḥ*, all of which are considered to be auxiliaries of the Vedas. In olden times, all these branches used to be studied by every vedic priest either after completing his studies of the Veda, or simultaneously along with it.

[4]Here, we would like to acknowledge our great indebtedness to Bibhutibhusan Datta for his authoritative work on the *Śulbasūtras* entitled "*The Science of the Śulba*". Much of our discussion follows the path trodden by Datta, and many of the translations, figures, and explanations in this paper are borrowed partly or in full from his masterly exposition.

[5]Sometimes also referred to as the *Baudhāyana-śulbasūtra*. In Sanskrit, the term Baudhāyana has the meaning "of Bodhāyana".

[6]This is the view maintained by scholars based on the archaic usages found in the text as well as the terse style of composition.

Table 1: Some topics covered in the *Bodhāyana-śulbasūtra*.

No.	Sanskrit name	English equivalent
1.	रेखामानपरिभाषा	Units of linear measurement
2.	चतुरश्रकरणोपायः	Methods of construction of squares and rectangles
3.	करण्यानयनम्	Obtaining surds / so-called Pythagorean theorem
4.	क्षेत्राकारपरिणामः	Transformation of shapes of areas (e.g. square to rectangle, trapezium or a circle, and vice versa)
5.	नानाविधवेदिविवरणम्	Plans for different sacrificial grounds (*dārśa*, *paśubandha*, *uttra*, *sutrāmaṇi*, *agniṣṭoma*, etc.)
6.	अग्नीनां प्रमाणक्षेत्रमानम्	Areas of the sacrificial fires/altars
7.	इष्टकासङ्ख्यापरिमाणादिकथनम्	Specifying the number of bricks used in the construction of altars including their sizes and shapes
8.	इष्टकोपधाने रीत्यादिनिर्णयः	Choosing clay, sand, etc., in making bricks
9.	इष्टकोपधानप्रकारः	Process of manufacturing the bricks
10.	श्येनचिदाद्याकारनिरूपणम्	Describing the shapes of *śyenaciti*, etc.

Dvārakānāthayajvā called *Śulbadīpikā*.

The *Āpastamba-śulbasūtra* consists of 223 *sūtras* divided into twenty one chapters, and has at least four well known commentaries by Kapardisvāmī, Karavindasvāmī, Sundararāja, and Gopāla. The *Kātyāyana-śulbasūtra* has two commentaries by Rāmacandra and Mahīdhara respectively.

It may be noted that all the topics discussed in the *Śulbasūtras* are in one way or another linked with the construction of sacrificial altars. This has been indicated by Bodhāyana right at the beginning of his eponymous *Bodhāyana-śulbasūtra* (*BoŚs* I.1):[7]

> **अथेमे अग्निचयाः।**
>
> Now [the construction of] altars associated with the [different sacrificial] fires [is presented] here.

Though the tradition of construction of altars should at least be traced back to the period of *Saṃhitās* and *Brāhmaṇās* (not later than 2500 BCE), some

[7] All the *sūtra* numbers in this paper are as per S. N. Sen and A. K. Bag, *The Śulbasūtras*.

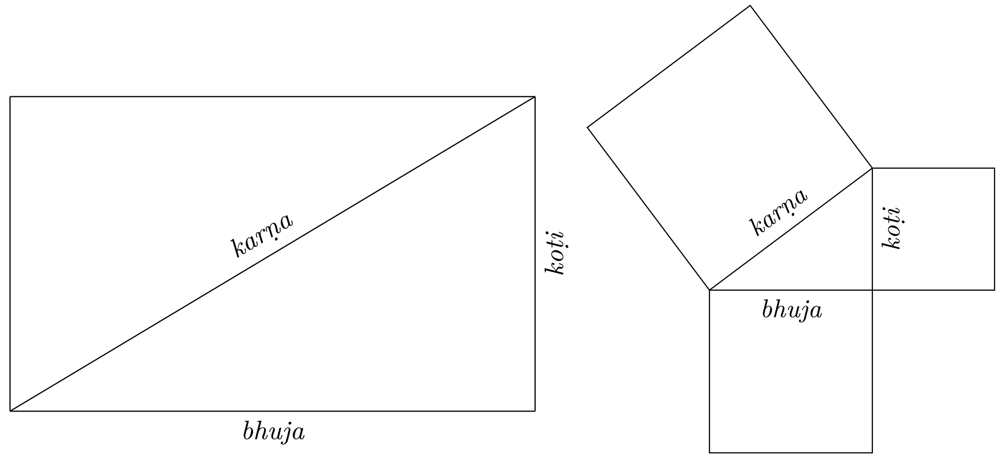


Figure 1: The so-called Pythagorean theorem as visualised in the *Śulbasūtras*.

scholars are of the view that the period of composition of the *Śulbasūtras* could be somewhere between 800–500 BCE.

10.3 Statement of the so-called Pythagorean theorem

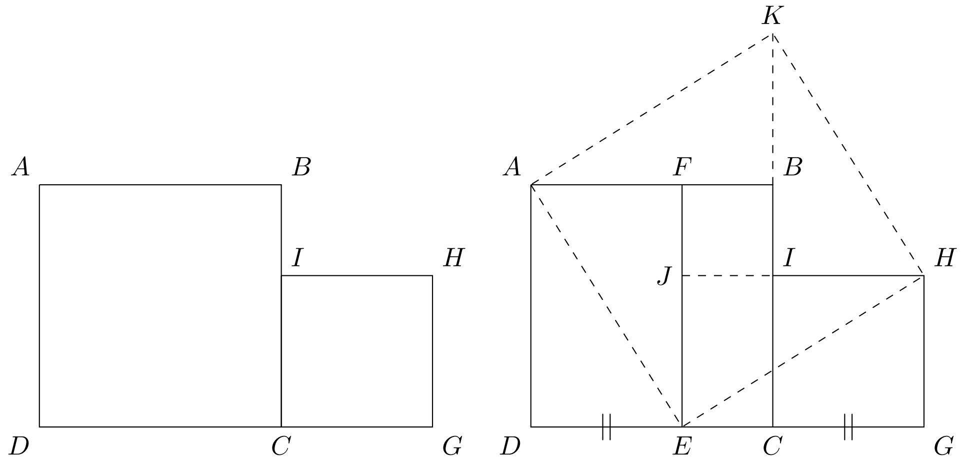
The actual enunciation of the so-called Pythagorean theorem in the *Śulbasūtrās* is not with respect to the right-angled triangle but with respect to the sides and diagonal of a rectangle or a square. For instance, the theorem given by Bodhāyana in his *Śulbasūtrā* is as follows (*BoŚs* I.48):

> **दीर्घचतुरश्रस्य अक्ष्णयारज्जुः पार्श्वमानी तिर्यङ्मानी च यत्पृथग्भूते कुरुतः तदुभयं करोति।**
>
> The diagonal of a rectangle produces both [areas] which its length and breadth produce separately.

Thus, Bodhāyana states that the area of the square produced by the diagonal of the rectangle is equal to the sum of the areas of the squares produced by the sides of the rectangle. The knowledge of this theorem is implicitly assumed in some of the subsequent methods discussed in the *Śulbasūtrās*. While the *śulbakāras* do not explicitly give any proof for this theorem, it is interesting to note that Bodhāyana records numerous 'Pythagorean' triplets in the very next *sūtra*. A subsequent procedure for constructing a square that is the sum of two unequal squares (see Section 10.4.1), and various other area preserving transformation techniques discussed later, strongly suggest that the *śulbakāras* knew the proof for the above theorem.

10.4 Methods for combining areas

The *Śulbasūtrās* discuss a number of techniques for combining the areas of figures including adding and subtracting squares, combinations of triangles and



(a) Original squares. (b) Combined square.

Figure 2: Construction of a square whose area is the sum of two unequal squares.

pentagons, etc. In this section, we discuss some of these important techniques.

10.4.1 Constructing a square that is the sum of unequal squares

To combine unequal squares, Bodhāyana, Āpastamba, and Kātyāyana essentially give the same method. The prescription given by Bodhāyana for this is as follows (*BoŚs* I.50):

> **नानाचतुरश्रे समस्यन् कनीयसः करण्या वर्षीयसो वृद्ध्रमुल्लिखेत्। वृद्ध्रस्य अक्ष्णयारज्जुः समस्तयोः पार्श्वमानी भवति।**
>
> Desirous of combining different squares, may you mark the rectangular portion of the larger [square] with a side (*karaṇyā*) of the smaller one (*kanīyasaḥ*). The diagonal (*akṣṇayārajjuḥ*) of this rectangle (*vṛddhra*) is the side (*pārśvamānī*) of the sum of the two [squares].

The method prescribed by Bodhāyana can be understood as follows. In Figure 2, $ABCD$ and $IHGC$ are the two unequal squares desired to be combined. The method outlined in the above *sūtra* prescribes to mark the points F and E on AB and DC such that $AF = DE = CG$, and then to draw the diagonal AE of the resulting rectangle $AFED$. It then clearly states that AE is the required side of the desired square whose area is equal to the sum of the two given squares.

To understand the rationale of this method, let us complete the desired square $AEHK$, denoted by the dashed lines in Figure 2b. It is evident from the figure that

$$\begin{aligned}\square ABCD + \square IHGC &= \triangle ADE + \triangle AFE + \triangle EJH + \triangle EGH + \square FBIJ \\ &= \triangle KIH + \triangle AFE + \triangle EJH + \triangle ABK + \square FBIJ \\ &= \square AEHK.\end{aligned}$$

Thus, we can see that the construction succinctly described in the above *sūtra*, though not explicitly stated as such, turns out to be an unambiguous proof of the so-called Pythagorean theorem. It may be noted that the *sūtra* style of composition was designed for easy and efficient transmission of knowledge orally, and thus the *sūtras* were brief and to the point.[8] Explicit proofs would likely have been taught directly by the teacher to the student, and would not have been reduced to the *sūtra* form due to their lengthy nature.

10.4.2 Constructing a square that is the difference of two squares

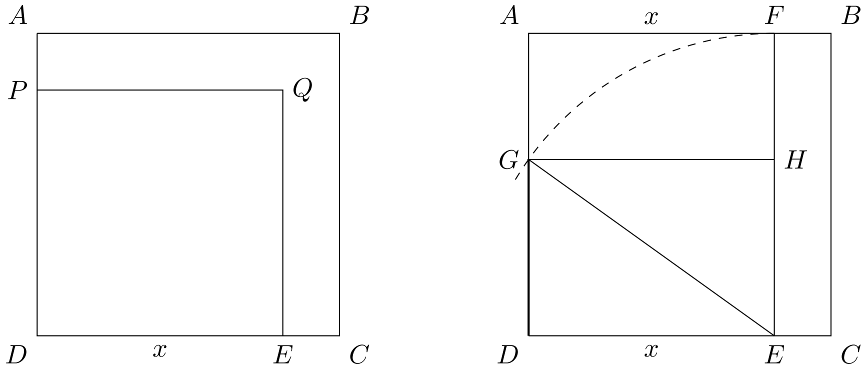
Bodhāyana states (*BoŚs* I.51):

> **चतुरश्राच्चतुरश्रं निर्जिहीर्षन् यावन्निर्जिहीर्षेत् तस्य करण्या वर्षीयसो वृद्ध्रमुल्लिखेत्। वृद्ध्रस्य पार्श्वमानीम् अक्ष्णया इतरत् पार्श्वमुपसंहरेत्। सा यत्र निपतेत् तदपच्छिन्द्यात्। छिन्नया निरस्तम्।**
>
> Wishing to deduct a square from a square, one should cut off a segment by the side of the square to be removed. One of the lateral sides of the segment is drawn diagonally across to touch the other lateral side. The portion of the side beyond this point should be cut off.

The given problem is to find out the side of that square whose area will be equal to the difference between the areas of the two given squares. We explain the stated procedure to obtain this with the help of Figure 3. Here, $ABCD$ is a large square, from which a square of side x is desired to be removed. To this end, mark F and E on AB and DC such that $AF = DE = x$. Join EF. Now, a rope whose length is equal to the side $AD = FE$ of the larger square is fixed at E and stretched and rotated to describe the arc of the circle denoted by the

[8] A *sūtra* has been traditionally defined as follows:

> अल्पाक्षरमसन्दिग्धं सारवद्विश्वतोमुखम् ।
> अस्तोभमनवद्यञ्च सूत्रं सूत्रविदो विदुः ॥
>
> Knowers of *sūtra* consider that as *sūtra* which is concise, unambiguous, full of essence, has universal applicability, non-redundant, and flawless.



(a) Squares to be subtracted.　　(b) Side of resulting square.

Figure 3: Determining the side of a square whose area is the difference of two squares.

dashed curve starting at F in the figure. This arc intersects AD at G. It can be seen from the right-angled triangle GDE that

$$GD^2 = GE^2 - DE^2 = AD^2 - x^2.$$

Thus, GD corresponds to the side of the square whose area is the difference of the two given squares.

10.4.3 Constructing a square whose area is n times a given square

The *Kātyāyana-śulbasūtra* (*KāŚs* 6.7) gives an interesting method for obtaining a square whose area is equal to the sum of the areas of a large number (say n) of squares:

> **यावत्प्रमाणानि समचतुरश्राणि एकीकर्तुं चिकीर्षेत्, एकोनानि तानि भवन्ति तिर्यक्।**
> **द्विगुणान्येकत एकाधिकानि। त्र्यस्रिर्भवति। तस्येषुस्तत्करोति।**
> As many squares [of equal size] as you wish to combine into one, the transverse line will be [equal to] one less than that; twice a side will be [equal to] one more than that; [thus] form a triangle (*tryasri*).[9] Its arrow (i.e., altitude) will do that.

Consider that there are n squares each of area a. It is desired that we obtain a square whose area is equal to the sum of the areas (i.e., na^2) of all the n squares. The procedure given by Kātyāyana is to construct an isosceles triangle, say ABC, whose base is of length $a(n-1)$ and sides are of length $\frac{a(n+1)}{2}$. It is said that the altitude of the triangle (AD) will give the side ($a\sqrt{n}$) of a square whose area will be na^2.

[9]The word *tryasri* literally means a three-sided figure.

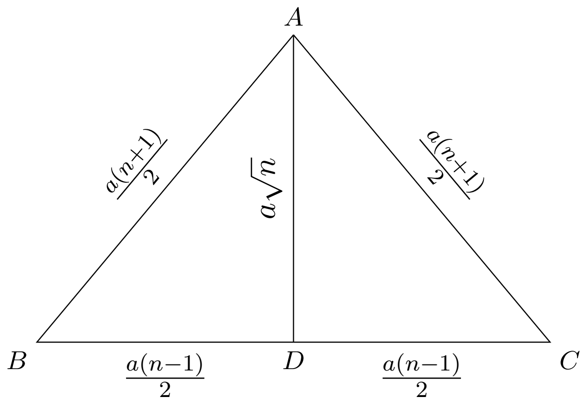


Figure 4: Kātyāyana's method for determining the side of a square whose area is n times a given square.

In Figure 4, AD is the altitude of the isosceles triangle ABC. Also, by construction,

$$AB = AC = \frac{a(n+1)}{2}, \qquad \text{and} \qquad BD = DC = \frac{a(n-1)}{2}.$$

Hence, considering the right-angled triangle ABD, we have

$$AD = \sqrt{AB^2 - BD^2} = a\sqrt{n}.$$

The prescription given above may look fairly straightforward and simple. But what is noteworthy is the amalgamation between geometry and algebra that is required in order to come up with this prescription in its most 'general' form.[10]

10.5 Methods for transforming figures

An interesting feature of the altars of different shapes described in the vedic texts is the requirement for them to have the same area for certain sacrifices. For instance, it is required that the three primary altars known as the *Gārhapatya*, the *Āhavanīya*, and the *Dakṣiṇa*, having the shapes of a circle, a square, and a semi-circle respectively, should all have the same area. Thus, the vedic priests had to deal with problems such as squaring a circle and vice-versa, transforming a rectangle into a square and vice-versa, etc. Some of the methods developed by the vedic priests to address these problems are discussed below.

[10]By 'general' form we mean the usage of *yāvat-tāvat* (as much-so much), which has been denoted by n in our explanation of the *sūtra*.

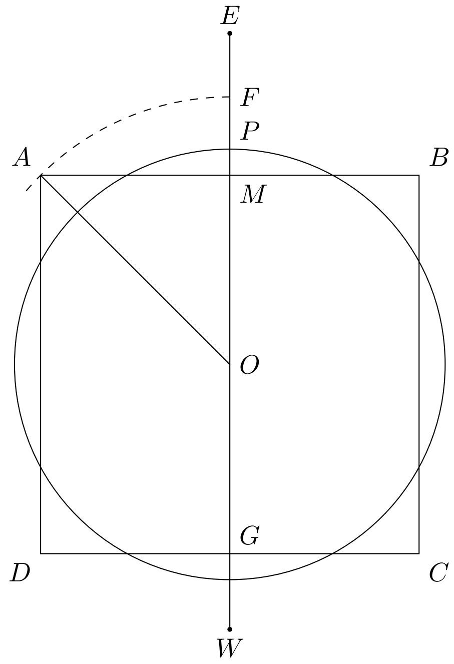


Figure 5: Transforming a square into a circle.

10.5.1 Transforming a square into a circle

The prescription given by Bodhāyana (*BoŚs* I.58) for transforming a square into a circle is as follows:

> **चतुरश्रं मण्डलं चिकीर्षन्, अक्ष्णयार्धं मध्यात् प्राचीम् अभ्यपातयेत्। यदतिशिष्यते तस्य सह तृतीयेन मण्डलं परिलिखेत्।**
>
> Desirous of transforming a square into a circle, may the [length of the] semi-diagonal (*akṣṇayārdhaṃ*) be marked along the east direction starting from the centre. Whatever portion extends [beyond the side of the square], by adding one-third of that [to the semi-side of the square] may the circle be drawn.

The given procedure for circling a square can be explained with the help of Figure 5. Here, $ABCD$ is a square with side $2a$. EW corresponds to the east-west line which vertically bisects the square. Let O be the centre of the square. Now, with O as centre and OA as radius, describe an arc which intersects EW at F. On the line segment MF, mark P such that $MP = \frac{1}{3}MF$. Now, to complete the procedure, we draw a circle with the centre O and radius OP.

The accuracy of this area preserving transformation can be estimated as follows. Mathematically, since $AB = 2a$, $OA = a\sqrt{2}$, and $MF = a(\sqrt{2} - 1)$, the radius of the desired circle ($OP = r$) is given by

$$r = a + \frac{a}{3}(\sqrt{2} - 1) = \frac{a}{3}(2 + \sqrt{2}).$$

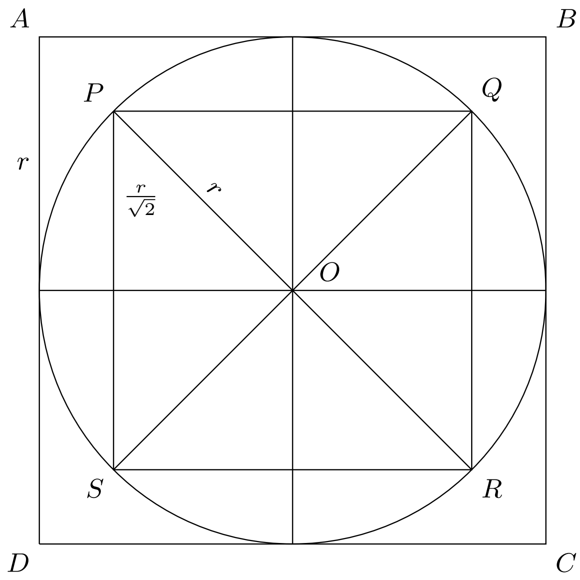


Figure 6: Transforming a circle into a square.

Taking the modern value of π, the area of the circle thus obtained is approximately equal to $4.069a^2$. This is an error of less than 2 percent versus the area $4a^2$ of the original square.

10.5.2 Transforming a circle into a square

The method proposed in the *Āpastamba-śulbasūtra* (*ĀpŚs* 3.3) to transform a circle into a square is as follows:

> **मण्डलं चतुरश्रं चिकीर्षन्, विष्कम्भं पञ्चदशभागान् कृत्वा द्वावुद्धरेत्। त्रयोदशावशिष्यन्ते। सा अनित्या चतुरश्रम्।**
>
> Wishing to convert a circle into a square, divide the diameter into fifteen parts and remove two of them. Thirteen [parts] will be remaining. This is the gross value of a side of the equivalent square.

As per the above *sūtra*, the side ($2a$) of a square having the same area as a given circle with diameter d is stated to be

$$2a = d - \frac{2}{15}d.$$

The rationale[11] behind this formula may be understood from Figure 6. The figure depicts a circle having radius r, a circumscribing square $ABCD$ having area $4r^2$, and an inscribed square $PQRS$ having the area $2r^2$. From the figure it can be easily seen that the area of the circle will be somewhere in between

[11] Based on the explanation given by Bibhutibhusan Datta, *The Science of the Śulba*, pp. 146–147.

these two quantities. If we approximate the area of the circle to be the mean of the areas of the two squares, we have

$$\text{Area of the circle} = \frac{4r^2 + 2r^2}{2} = 3r^2.$$

If $2a$ be the side of the square having the same area as the circle, then, $4a^2 = 3r^2$ or

$$2a = \sqrt{3}r.$$

Datta then shows[12] through a somewhat involved process that $\sqrt{3}$ can be approximated as

$$\sqrt{3} \approx 1 + \frac{2}{3} + \frac{1}{15} = \frac{26}{15}.$$

Substituting this value in the above equation, we have

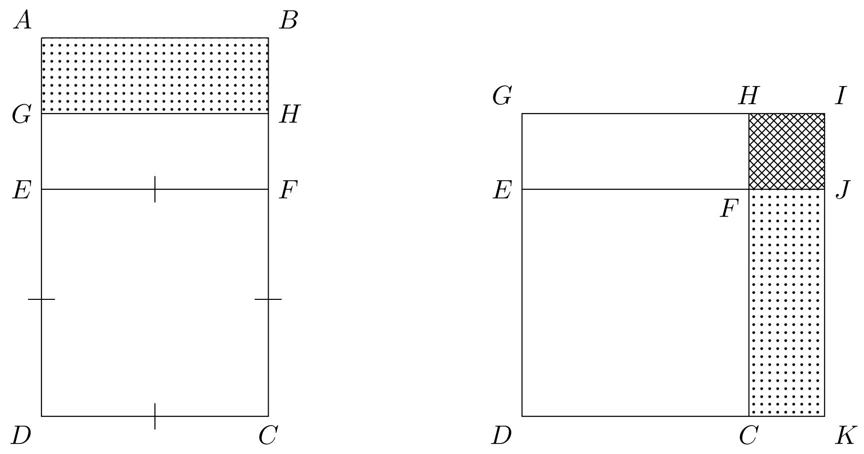
$$2a = \frac{26}{15}r = \frac{13}{15}d = d - \frac{2}{15}d.$$

10.5.3 Transforming a rectangle into a square

Āpastamba's rule (*ĀpŚs* 2.7) for transforming a rectangle into a square is as follows:

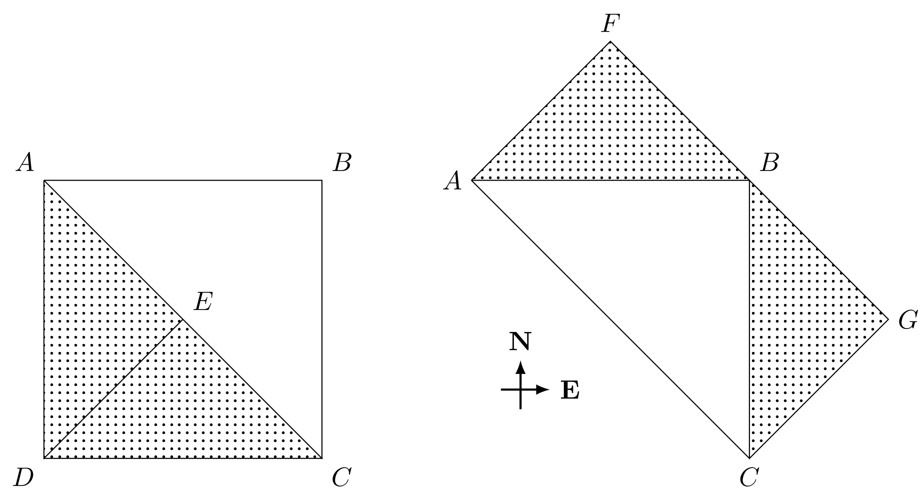
> **दीर्घचतुरश्रं समचतुरश्रं चिकीर्षन्, तिर्यङ्मान्या अपच्छिद्य शेषं विभज्य उभयत उपदध्यात्। खण्डं आगन्तुना संपूरयेत्। तस्य निर्हारः उक्तः।**
>
> Wishing to turn a rectangle into a square, one should cut off a part equal to the transverse side and the remainder should be divided into two and juxtaposed at the two sides [of the first segment]. The bit [at the corner] should be filled in by an imported bit. The [procedure for] expulsion of this was stated earlier.

Consider that the rectangle $ABCD$ depicted in Figure 7a is to be converted into a square. To this end, cut off ED and FC equal to the transverse side CD so that $EFCD$ is a square. Now, the remainder $ABFE$ is divided into half and the upper portion is rotated and placed to the side of FC (see Figure 7b). Complete the larger square $GIKD$ by adding a square $HIJF$ to fill up at the corner. From the figure it can be observed that the area of the rectangle $ABCD$ is equal to the difference of the areas of the squares $GIKD$ and $HIJF$. The square whose area is equal to the difference between the areas of these two squares can be determined using the method described in Section 10.4.2. This is what is stated in the last part of the *sūtra* through the phrase *tasya nirhāraḥ uktaḥ*.



(a) Original rectangle $ABCD$. (b) Rearranged figure with $HIJF$ filled.

Figure 7: Transforming a rectangle into a square.



(a) Original square $ABCD$. (b) Rectangle $AFGC$ of same area.

Figure 8: Transforming a square into a rectangle.

10.5.4 Transforming a square into a rectangle

The *Kātyāyana-śulbasūtra* (*KāŚs* 3.4) describes the following method to transform a square into a rectangle:

> **समचतुरश्रं दीर्घचतुरश्रं चिकीर्षन्, मध्येऽक्ष्णया अपच्छिद्य तच्च विभज्य अन्यतरत् पुरस्तात् उत्तरतश्च उपदध्यात्।**
>
> Wishing to convert a square into a rectangle, one should cut diagonally in the middle, divide one part again and place the two halves to the north and east of the other part.

In Figure 8, let $ABCD$ be the square given. As per the prescription given in

[12]See Bibhutibhusan Datta, *The Science of the Śulba*, pp. 194–195.

the *sūtra*, this square has to be first cut diagonally along AC. Now, the triangle ADC has to be further divided into two halves by dropping the perpendicular DE on AC. The two halves represented by the triangles AED and CED are now placed on the other side of the square to form the triangles AFB and CGB respectively. We thus obtain the rectangle $AFGC$ which has the same area as the original square $ABCD$.

The dimensions of the rectangle obtained by this method are pre-determined by the dimensions of the original square. Elsewhere,[13] the *Śulbasūtras* also discuss means of converting a square into a rectangle having a desired side.

10.6 Concluding remarks

In this paper, we have given a brief introduction to the *Śulbasūtras*, and discussed some of the geometrical techniques described in this text. In the *Śulbasūtras* we find the earliest expression of the so-called Pythagorean theorem, a few centuries before the Greek philosopher and mathematician Pythagoras (c. 500 BCE). Several geometric constructions described in this text reveal a deep understanding of this theorem and strongly suggest the knowledge of its proof among the vedic priests.

The ancients showcased remarkable dexterity and geometric skill in adding and subtracting areas of different figures into a single figure. In this process, they also developed clever procedures to calculate and measure the values of surds to a great degree of accuracy, the details of which will be discussed in a later paper. The ancients also did not shy away from tough exercises such as squaring a circle and circling a square, problems which have troubled the best minds in the history of humanity. It is remarkable that their method of circling a square has an error of less than two percent. They also developed simple and intuitive techniques to convert a rectangle into a square and vice-versa.

We hope that this exposition brings to light an important aspect of the history of mathematics in India, and inspires teachers to introduce some of the methods and techniques of Indian mathematics in general, and the *Śulbasūtras* in particular, in the school classroom. The ideas and concepts presented in these texts, composed almost 2500 years ago, portray the scientific legacy of the remote past, and it is thus crucial for Indian students to have a chance to learn more about their own scientific heritage. In a subsequent paper, we plan to discuss in detail the treatment of surds, and other interesting topics found in the *Śulbasūtras*.

[13] For instance, see *ĀpŚs* 3.1 or *BoŚs* I.53, and the discussion by Bibhutibhusan Datta, *The Science of the Śulba*, pp. 85–90.

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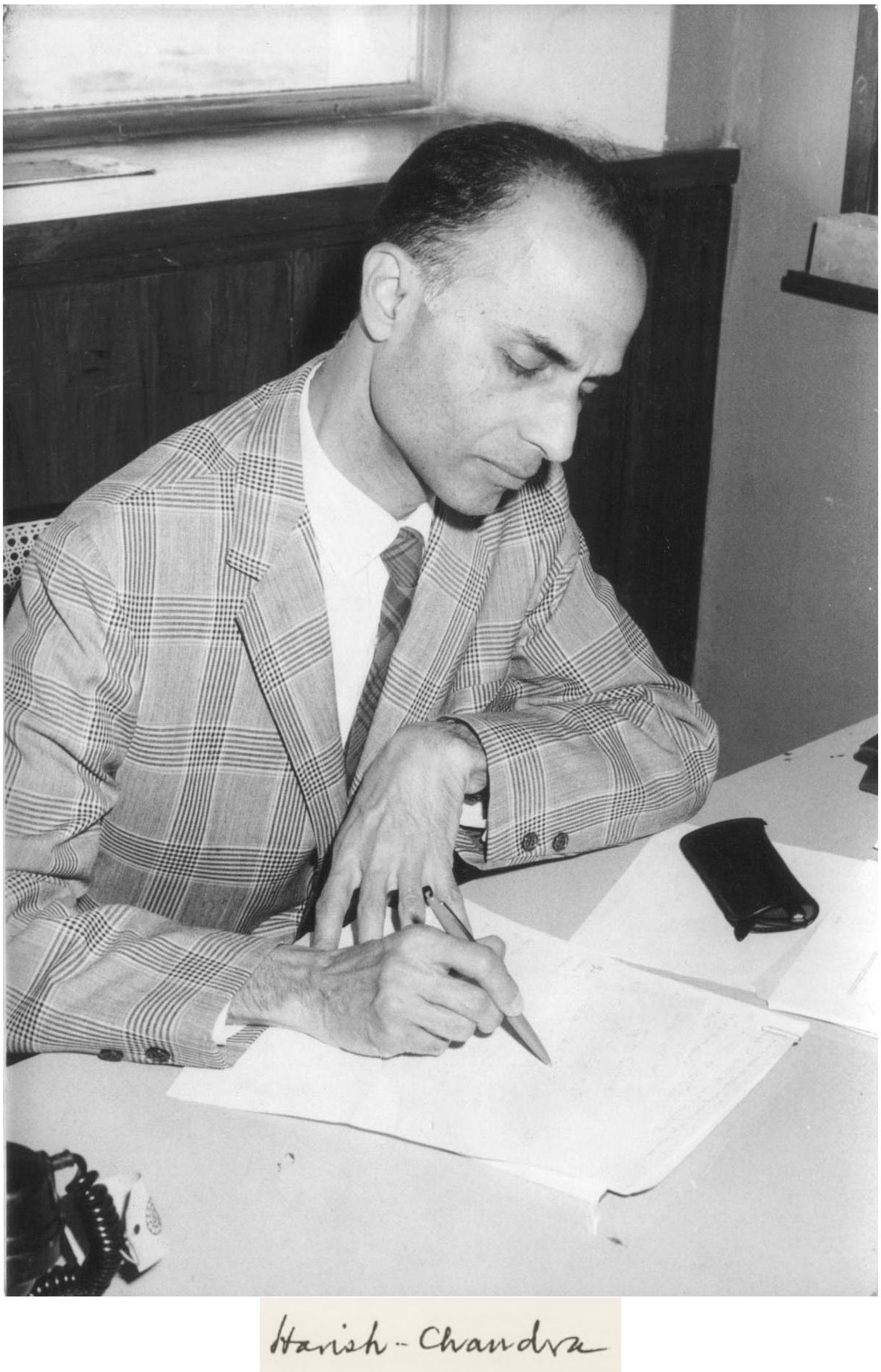
11 **Harish-Chandra** *by R.P. Langlands F.R.S.*



Harish-Chandra (11 October 1923 — 16 October 1983)[1]

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[1]Elected F.R.S. 1973

Harish-Chandra was one of the outstanding mathematicians of his generation, an algebraist and analyst, and one of those responsible for transforming infinite-dimensional group representation theory from a modest topic on the periphery of mathematics and physics into a major field central to contemporary mathematics.

Kanpur and Allahabad

He was born on 11 October 1923 in Kanpur in North India. His paternal grandfather had been a senior railroad clerk in Ajmer who, to finance his son's education, had resigned his post to collect the lump sum given as severance pay, and then rejoined the railroad, his seniority lost, in a junior position. His son, Chandrakishore, later the father of Harish-Chandra , had gained admission to the highly selective Thomason Engineering College at Roorkee, which had been founded by Dalhousie in 1857, and which was responsible for the training of civil engineers for the department of public works. Every graduate was assured a position in the government services and admission was much coveted.

Harish-Chandra's father, a civil engineer, eventually rose quite high, reaching the middle echelons of the Indian Service of Engineers, and retiring as Executive Engineer of the Uttar Pradesh Irrigation Works; but his early career would have been spent in the field, usually on horseback, inspecting and maintaining the dikes of the extensive network of canals in the northern plains. Roorkee College and the effort of competing with the British on still unfamiliar technical ground seem to have produced a breed of serious-minded, conscientious men, devoted to their work and somewhat distant from their families. None the less, Chandrakishore's family did share the life of the canal posts, and Harish-Chandra, although not a robust child, often accompanied his father on his rounds, but it was not until later, when he was a young man and his father retired, that they became close.

In 1937, just three years before the father's retirement, the whole family was able for the first time to take an extended vacation. They travelled to Kashmir, their baggage carried by seventeen porters. Harish-Chandra, who remained a keen walker throughout his life, always recalled with pleasure the hikes in the hills with his father. Later on, after Harish-Chandra's moves to Bangalore and Cambridge, they corresponded regularly, and his respect for the high-minded, religious Chandrakishore was to be an abiding influence on Harish-Chandra.

His life as a child was divided between the canal posts and the home of his maternal grandfather in Kanpur. His mother, Satyagati Seth or, after her marriage, Chandrarani, was the daughter of a lawyer, Ram Sanehi Seth. Both he and his wife were descendants of old *zamindari* families, feudal landowners, in what is now Uttar Pradesh. One branch of the Seths appears in the 18th century as the proprietors of an important banking house who came to grief in the struggle between the East India Company and the Newab of Bengal. A more recent incident, still recounted in the family, is that it was able to offer refuge to the highspirited but ill-fated Rani of Jhansi during the Mutiny of 1857, in which she was a central figure. As a token of gratitude she left behind a sword. Since the family is of the Rajput or Khatri caste, the men still have occasion to don it at their wedding celebrations. Harish-Chandra might have worn it as well if he had not, to the scandal of his family, insisted on a civil ceremony.

Although related to a prominent family Ram Sanehi Seth achieved his substantial social position through his own efforts, for his immediate family was of modest means, and his large house, in which Harish-Chandra was to pass much of his adolescence, was home to innumerable relatives. Precocious in his studies and often ill, Harish-Chandra did not find the turbulent atmosphere congenial, and one suspects that both home and school, where he was teased by older, rougher classmates, exacerbated an innate timidity.

However, in his grandfather's home, as in many North Indian households, music was cultivated, and Harish-Chandra took from it a love of music which he never lost, not only for the *ragas* of his homeland, but also in later life for composers of the West, above all Beethoven.

Chandrarani appears to have inherited the energy and ambition of her father and to have passed it on to her children, all of whom had distinguished careers. Satish, older than Harish by seven years, entered in 1939 the Indian Civil Service, the élite administrative corps of Imperial India, and became ultimately, after Independence, Chief Secretary of Uttar Pradesh and then Secretary to the Government of India in the Ministry of Defence and Supplies during the Indo-Pakistani War. Suresh, younger than Harish by seven years, was an engineer with the Indian Railways, and then joined the State Corporation of India. He is now executive director of a private corporation. The sole daughter, Vimala, married Jagdish Behari Tandon who, having served with the Indian Agency in Burma, where he was made a prisoner-of-war, joined the Indian Administrative Service, the successor to the Indian Civil Service, retiring as a member of the Board of Revenues of the Uttar Pradesh Government. The husband of a cousin was an admiral in the Indian Navy.

Considerable attention was given to the early education of Harish-Chandra. A tutor was hired, and there were visits from a dancing master and a music master. At the age of nine he was enrolled, younger than his schoolmates, in the seventh class. He completed Christ Church High School at fourteen, and remained in Kanpur for intermediate college, which he finished at sixteen, and then matriculated at the University of Allahabad, where he obtained the B.Sc. in 1941 and the M.Sc. in 1943 at the age of twenty.

High-strung and frequently ill, Harish-Chandra was especially vulnerable at the time of examinations, all of which he seemed to take while suffering from some malady, serious or comic, from paratyphoid to measles. This did not prevent him from performing brilliantly. For the M.Sc., when he was examined by the physicist C. V. Raman, F.R.S., he was the first in the state of Uttar Pradesh, receiving 100% on the written test.

He had learned some mathematics, as far as the calculus, and some science from his father's textbooks but his introduction to modern science came at the university. He described many years later how Dirac's *Principles of quantum mechanics*, which he had discovered in the university library in 1940, evoked in him the desire to devote his life to theoretical physics. Two years later K. S. Krishnan, F.R.S., an excellent physicist and a widely cultivated man, was appointed Professor of Physics in Allahabad. He encouraged Harish-Chandra in every possible way, lending him books like Hermann Weyl's *Raum-Zeit-Materie* and recommending him as a research student in physics to H. J. Bhabha, F.R.S., at the Indian Institute of Science in Bangalore. The mild-mannered, gentle Krishnan inspired in Harish-Chandra not only respect but also an affection that never abated. For the boisterous, egoistical Raman and his achievements he had also, in spite of the difference in their temperaments, a high regard, but his own ascetic nature did not allow him to perceive the virtues accompanying the high-living Bhabha's extravagance.

Bangalore and Cambridge

The South Indian environment would have been foreign to Harish-Chandra, but he spent the first six months lodging with old friends from Allahabad, Mrs H. Kale, who had been his French teacher at the university, and her husband Dr G. T. Kale, a botanist who had moved to Bangalore to take up duties as librarian at the Institute.

The eager, serious student was an inviting target for the pranks of their young daughter, Lalitha, but the interruptions could not have been entirely unwelcome, for many years later, when he returned to India on a visit, she, now a strikingly beautiful young woman, became his wife. There were other interruptions. Raman, already fifty-five, had taken a liking to Harish-Chandra and would drop by unexpectedly to invite him for a walk. Harish-Chandra would also walk alone, sometimes with his sketchbook in hand, for at that time he liked to draw and to paint. He was an excellent copyist. He later gave up painting completely, although in 1951 when visa difficulties prevented him from travelling he, in his own words, made a virtue of necessity and enrolled in a painting course in the Summer School at Columbia University. A few sketches, made on vacation, remain in the family, as well as a copy of Rubens's *Le Chapeau de Paille* from the time in Allahabad, treasured by his mother-in-law, Mrs Kale. He copied it for her as an eighteen-year-old in a gesture of affection and gratitude to a favourite teacher from a collection of reproductions of paintings from the National Gallery with which his father had presented him, choosing it as much for its French title as for any artistic reason.

Shortly before leaving Columbia, in an interview with the alumni magazine he tried to express his mathematical aesthetics in a metaphor from painting, stating that 'In mathematics there is an empty canvas before you which can be filled without reference to external reality.' In the final phrase he is thinking perhaps more of mathematics than of painting for he adds, 'The only value of mathematics lies in its internal structure.' This is an extreme view, but it has real validity if taken to refer to his own style and to express his satisfaction at having found in mathematics a subject better suited to his own inclinations than the physics he had abandoned because 'it is basically an empirical science.'

In painting, as in other things, he admired excellence. He was especially fond of the Impressionists and in his last year, often too ill to work, he spent many hours with reproductions of the paintings of Cézanne and van Gogh, reflecting on their lives, and perhaps seeing in their intensity and struggles a similarity to himself.

Gandhi's *Quit India* movement had been broken in 1942, and from then until the end of the war the independence movement was dormant. So Harish-Chandra's time in Bangalore was untroubled by politics. Indeed, although his parents had been supporters of Gandhi, his father adopting the wearing of khadi, Harish was never more than superficially touched by politics. He had strong views, which he would sometimes vehemently defend, but he was not distracted by them, and was impatient with the hypocrisy and sentimentality, perhaps simply with the welter of emotions, that politics by their very nature entail.

Harish-Chandra's career as a physicist was to be brief–two years in Bangalore with Bhabha and two years in Cambridge with P. A. M. Dirac, F.R.S. He himself does not appear to have attached much importance to the work done then, but it is of biographical interest and does occupy considerable space in his *Collected works.*

In Bangalore there were two themes, both reflecting concerns of Bhabha and indirectly Dirac. The first, on which he wrote some papers alone and some with Bhabha, was classical point-particles, their equations of motion, and the fields associated with them. Its origins lie in a 1938 paper by Dirac in which he derived equations of motion for a classical charged point-particle moving in an external field by examining the combined effects of the external field and the field of the particle itself on a small tube surrounding the world-line of the particle. He lets the diameter of the tube go to zero, keeping only the finite part of the energy and momentum communicated to the tube, and obtains equations agreeing with those of the Lorentz theory. Similar ideas can be applied to other point-particles and the associated fields, and Bhabha and Harish-Chandra developed them extensively, especially for neutrons and their classical meson fields. This work found no echo in the literature.

The second theme, relativistic wave equations, especially for particles of higher spin, touches issues that, although somewhat peripheral, remained of concern to mathematical physicists and are still not completely resolved. It deals with problems that in the 1940s were largely algebraic and some of the papers, like those on the Dirac matrices and those on the Duffin-Kemmer matrices, are purely so. The innate algebraic facility displayed in them, and in the early Princeton papers, was transformed by experience and effort into the powerful technical skill of the papers on representation theory. As it gained in strength it lost in ease but never in resourcefulness.

Serious problems arise when attempting to construct a theory of elementary particles with higher spin. The inconsistencies that arise in attempts to include the affect of an external field appear to be the most vexing. Harish-Chandra alludes to this problem and even suggests, in an appropriately tentative introduction to one of his three papers on the topics, that his efforts might lead to its solution, but sets himself a more modest goal.

Apparently there are several desirable features for a relativistic wave equation in addition to Lorentz invariance: (i) unique rest mass; (ii) unique spin; (iii) positivity of total energy or total charge. All these requirements were met by the Dirac–Fierz–Pauli theory but at the cost of a simple Lagrangian formulation. Harish-Chandra attempted to preserve the simple Lagrangian, the unique rest mass that he took to be greater than zero, and the positivity, without which there is no quantization, but to abandon the unique spin. He was then able, among other things, to construct a formalism for elementary particles with spin that could take on both values, $\frac{3}{2}$ and $\frac{1}{2}$. However, I understand that nowadays, when there is a great variety of particles with higher spin whose existence has been experimentally discovered, the problems appear in a much different light than they did forty years ago. Either they are dealt with in the context of supersymmetry, where the inconsistencies can by a felicitous choice of coupling constants be made to disappear, or the particles are treated as composites or resonances.

The earliest papers on point-particles had been communicated by Bhabha to the *Proceedings of the Royal Society* and had gained for Harish-Chandra not only the hyphen in his name, which was first placed there by a copy editor and which he decided to retain, but also the attention of Dirac, who had been requested by Bhabha as a special favour, the wartime mail between India and England being extremely slow, to correct proofs. On the basis of this work and perhaps recommendations from Bhabha as well, Harish-Chandra had been accepted by Dirac as a research student. Not long after the war in Europe had ended he set sail for England, and was on board ship when the atomic bomb fell on Hiroshima on 6 August 1945. Cambridge had still not returned to normal and was almost deserted when he arrived to take up residence in Gonville and Caius College.

In Cambridge his personal contacts with Dirac were infrequent. He attended his lectures at first but dropped out when he discovered that they were almost the same as the book. However, he did attend the weekly colloquium run by Dirac. He found that 'he was very gentle and kind and yet rather aloof and distant' and felt that 'I should not bother him too much and went to see him about once each term'.

The work on equations of particles with higher spin belongs on the whole to the Cambridge period, but his thesis proper was on a different, although closely related, topic: the classification of irreducible representations of the Lorentz group. It was proposed by Dirac and, as Harish-Chandra later remarked, was how he got started in group representations. They were to be his life.

One of the first papers on infinite-dimensional irreducible representations had been written by Dirac himself in 1944. He introduced it with the remarks:

> 'The Lorentz group is the group of linear transformations of four real variables ξ_0, ξ_1, ξ_2, ξ_3, such that $\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2$ is invariant. The finite representations of the group ...are all well known and are dealt with by the usual tensor analysis and its extension spinor analysis. None of them is unitary. The group has also some infinite representations which are unitary. These do not seem to have been studied much, in spite of their possible importance for physical applications.'

This is as close as one comes to the source of the theory of infinite-dimensional representations of semisimple and reductive groups, which as it turned out were to be of limited physical significance but of great mathematical import. Soon after Dirac's initial article three papers were written classifying the irreducible representations of the homogeneous Lorentz group, one by Harish-Chandra, who solved the problem posed by Dirac, another by Bargmann in the U.S.A. and a third by Gelfand–Naimark in the Soviet Union. The paper by Bargmann was the most influential of the three. He considered not only the usual Lorentz group defined by four-dimensional space-time but also the analogous group defined by two space dimensions and one time dimension whose representations have, surprisingly, a more complex structure, containing the discrete series of square-integrable representations, which would become, after Harish-Chandra had demonstrated their importance in general, the rogue's yarn running through the subject.

Harish-Chandra's own paper suffered from a lack of the rigour appropriate to the treatment of a topic in pure mathematics. He later commented that,

> 'Soon after coming to Princeton I became aware that my work on the Lorentz group was based on somewhat shaky arguments. I had naively manipulated unbounded operators without paying any attention to their domains of definition. I once complained to Dirac about the fact that my proofs were not rigorous and he replied, "I am not interested in proofs but only in what nature does." This remark confirmed my growing conviction that I did not have the mysterious sixth sense which one needs in order to succeed in physics and I soon decided to move over to mathematics.'

In fact, it seems that he had been preparing for the move for some time. In Cambridge he attended the lectures of J. E. Littlewood and Philip Hall, discovering mathematics as he began to doubt his vocation as a physicist. In Princeton he would write one more paper on physics and then, for all professional purposes, abandon the subject entirely. Harish-Chandra's nature was unrelentingly intense, and demanded a sharp focus. His daily and yearly routines grew simpler with the years, and his temperament became more ascetic and more impatient with richness of detail or complexity of character. The best aroused admiration and respect; to the rest he was almost ruthlessly indifferent. From himself, too, he demanded the best and had, I would guess from a distance, as a young man considerable confidence in his ability to achieve it. But he would not dabble.

So once he recognized that his talents lay elsewhere his active interest in physics ceased. However, the respect he continued to have for the subject and those whom he regarded as its greatest practitioners, above all Dirac, was enormous, amounting almost to a religious awe. He never accorded so much even to those mathematicians he most admired and was most eager to emulate, certainly not to himself or his own work.

Although he was convinced that the mathematician's very mode of thought prevented him from comprehending the essence of theoretical physics, where, he felt, deep intuition and not logic prevailed, and skeptical of any mathematician who presumed to attempt to understand it, he was even more impatient with those mathematicians in whom a sympathy for theoretical physics was lacking, a failing he attributed in particular to the French school of the 1950s.

Harish-Chandra was introspective and often reflected on his own working methods. At a conference in honour of Dirac shortly before his own death he expressed his views on the role of intuition.

> 'I have often pondered over the roles of knowledge or experience, on the one hand, and imagination or intuition, on the other, in the process of discovery. I believe that there is a certain fundamental conflict between the two and knowledge, by advocating caution, tends to inhibit the flight of imagination. Therefore a certain naïveté, unburdened by conventional wisdom, can sometimes be a positive asset.'

These remarks refer to Dirac but also, and quite consciously, to himself. However, his admiration of Kodaira and Siegel expressed perhaps a clearer assessment of his own gifts. Harish-Chandra came to mathematics relatively late and, in spite of enthusiastic initial attempts, there were broad domains of mathematics that he never assimilated in any serious way, although he learned all that he needed, which was considerable. None the less, for a mathematician of his stature and ambitions his base was narrow. He knew this; it troubled him; and he was more than a little defensive. In the circles in which he often found himself he had to be. However, what saved him was not, in my view, his intuition, of which he had relatively little, either geometric or algebraic, but an analytic power and algebraic facility unsurpassed in my experience. He of course exploited the ideas of others and techniques that were at hand—they were occasionally crucial—but by and large it is not too much of an exaggeration to say that he manufactured his own tools as the need arose, and that one of the grand mathematical theories of this century has been constructed with the skills with which one leaves a course in advanced calculus.

Over the years he kept himself informed in a casual way of developments in physics through popular articles and conversation, and it gave him great pleasure that the elder of his two daughters, Premala, chose to study it in graduate school. In his last years, no longer able to work long hours, he had more time to spend with her and his younger daughter Devaki. He would often thumb through Premi's textbooks and when she was home spent many hours closeted with her in his study to discuss all that she had learned since her last visit. That it was solid-state physics, with which he was unfamiliar, and not his first passion, elementary particle physics, only heightened the interest for him.

Princeton, Cambridge (Massachusetts), New York

In 1947-8 Dirac was a visiting professor at the Institute for Advanced Study in Princeton and Harish-Chandra was appointed his assistant. He remained a second year on his own. In Princeton he wrote one more paper on physics, became closer to Dirac and his family, and also formed a friendship with W. E. Pauli, F.R.S., and his wife. He had already met Pauli, first in England, probably at the International Conference on Fundamental Particles and Low Temperatures held at the Cavendish Laboratory in 1946. There, to Pauli's annoyance, he had had the temerity to suggest, correctly as it turned out, in a remark at the end of Pauli's lecture that Pauli had made a mistake. Later in Zurich, while Harish-Chandra was there for the summer to study German, Pauli had taken the occasion to invite him to his home. Above all, during his stay at the Institute, he plunged into mathematics, keen, to judge from his letters of the period, to master it all, and struck up friendships with other young visitors, for example F. Mautner, G. D. Mostow and I. Segal. Some were to last a lifetime.

Curiously enough the last paper on physics (1948b)[2], which dealt apparently with a mathematically well-defined problem suggested by a paper by Dirac is vitiated by a topological error. Otherwise Harish-Chandra might have anticipated by thirty years results of at least some speculative interest. He considers the motion of an electron in the field of a magnetic monopole, and professes to prove there are no bound states. It turns out there is one. So far as I can see Harish-Chandra does not observe that the eigenvalue problem he is solving is not for functions but for sections of a bundle, a point that Dirac in his own manner stressed. So Harish-Chandra goes astray when separating variables, is able to imitate the relativistic treatment of the hydrogen atom, and misses the novel feature of his equation.

At the same time the mathematical springs, having finally forced their way to the surface, rushed forth in a torrent that was not to abate for two decades, and indeed continued only slightly diminished until his death.

His letters of the time reveal an eager, confident, almost brash young man. In Princeton he took courses from C. Chevalley and E. Artin, but was disappointed in Hermann Weyl, F.R.S., whose personality was perhaps too elaborate for him.

[2]Numbers given in this form refer to entries in the bibliography at the end of the text.

At the suggestion of I. Segal he read Weil's book *L'intégration dans les groupes topologiques*. He read it carefully and quickly, immediately noticing a gap in the proof of the duality theorem. In 1949-50 he spent a year at Harvard as a Jewett Fellow to study algebraic geometry with O. Zariski with whom he seems to have got along well, but his fellowship was not renewed. Perhaps he spent too much time on representation theory, but he did learn some algebraic geometry, although little appears in his papers.

One incident is worth recounting because it shows an aspect of Harish-Chandra that was afterwards suppressed. Irving Segal was teaching a course on elementary number theory at Columbia in 1953-54 from T. Nagell's text, and one day was questioned by his students about a seemingly easy problem he had assigned, namely, to show the existence for all primes p not dividing $7abc$ of solutions of the congruence $ax^3 + by^3 \equiv c \pmod{p}$. He struggled with it for the rest of the hour but was stumped. He left, promising to return to the next meeting with a solution, and worked overnight on the problem, but without success. In desperation as the class approached, he consulted the many eminent specialists among his colleagues, but to no avail. He also mentioned the problem casually to Harish-Chandra, not expecting help, but the next day Harish-Chandra observed that the genus of the projective curve being one and there being at most 3 points at infinity the existence of solutions was for $p > 7$ a consequence of the Weil theorem. For 7 it can of course be verified directly. Later appeals to E. Artin and to Nagell himself yielded more elementary solutions but none simpler.

Algebraic geometry was not all he intended to learn. In a letter to Segal of 1951, in which he first describes his results on representation theory and then expresses his regrets at not being able to travel, and indicates his intention to take a course in painting during the summer, he goes on to say that he is to lecture on topological groups in the coming year and is beginning to be attracted to classical analysis, especially function theory (although he realizes it is somewhat unpopular with the modern young men) and the theory of modular functions. He describes his plans to spend most of his time on function theory and algebraic geometry, but in the next line expresses the pleasure with which he anticipates a course of Chevalley on class-field theory and zeta-functions. Turning to mathematical gossip, he mentions the 'sensational new developments in France concerning homotopy groups', noting with approval that 'Cartan certainly seems to have brought in some fresh blood into topology,' and observes that 'Chevalley is still busy with his exceptional groups,' adding that 'I am sure that after Elie Cartan he is the man who knows most about them.' Finally he reports on the work of G. Racah on the invariants of the exceptional groups, with its application to the calculation of their Betti numbers, observing in a somewhat patronizing tone, 'For a man who is not a professional mathematician, he seems to be exceptionally well informed about groups and is undoubtedly very able.' The mature Harish-Chandra was more focused, and more subdued.

In 1950 he took up a position at Columbia University in New York and remained there until 1963, although he spent several of the intervening years abroad. The academic year 1952-53 he spent in Bombay at the Tata Institute, returning also to Bangalore, where he was met at the airport by Raman. There he was married to Lalitha Kale, now Lily Harish-Chandra, who with good spirits, generous affection, patience, and all-round competence was to pamper him for thirty years.

It is not clear whether he seriously contemplated remaining in India. Bhabha seems to have made an effort to create a reasonable permanent position in Bombay, but whether Harish-Chandra felt that the proper recognition or the proper working conditions could only be obtained abroad, he did not stay and did not return again except for brief visits. He continued to concern himself with Indian mathematics and mathematicians. The strength of his influence can be measured by the number of Indians who have contributed in recent years to representation theory and related domains.

The academic year 1955-56 was spent at the Institute for Advanced Study and 1957-58 was spent in Paris on a Guggenheim Fellowship. He and Lily lived at Sceaux, where they were often visited by Andre Weil. Both he and Harish-Chandra were keen walkers, and would stroll in the nearby Parc de Sceaux, accompanied by Lily, who was usually ignored.

An incidental windfall from the time in Paris was a substantial increase in salary at Columbia. The dean at Columbia, unhappy with two leaves of absence so close together, expressed his displeasure in letters to Harish who was already in Paris and who, anxious to remain there, turned to Weil for counsel. Weil, for whom dean-baiting was an agreeable diversion, was glad to provide it and pointed out to Harish-Chandra that his scientific contributions of the preceding few years had been such that a large number of universities would be glad to have him, if Columbia felt it preferred to do without his services, and indeed at a considerably higher salary, for at the time Harish-Chandra's salary was on the low side. Weil urged him to communicate all this to the dean, and even dictated an appropriate letter. I am assured that it was sent, although reluctantly. The leave was certainly extended and his salary raised.

In Paris Harish-Chandra attended lectures by Weil on discrete groups and was fired with the ambition to prove the finiteness of the volume of the fundamental domain for arbitrary arithmetic subgroups of semisimple Lie groups.

Returning to Columbia he studied with great attention the papers of Siegel on reduction theory, working through the proofs repeatedly until towards 1960 he found the key to the general theorem. It belonged to a line of development that had felt the hand of many of the masters of number theory, from Gauss to Siegel, and it brought Harish-Chandra great satisfaction.

In a lecture delivered in 1955 entitled *On the characters of a semisimple Lie group* (1955b) Harish-Chandra was already attempting to discover the properties of the distribution character of an irreducible representation, whose existence he had established in 1952. By the time of his visit to Paris he was trying to show that it was in fact given by a locally integrable function on the group. This he first announced in 1963, but at least one essential feature of the proof was already in his mind in 1958, the reduction of the theorem on the group to a similar theorem on the Lie algebra.

If ϕ is a function on an appropriate invariant neighbourhood U of 0 in the Lie algebra then

$$f_\phi(\exp X) = |\det\{(\exp(\operatorname{ad} X/2) - \exp(-\operatorname{ad} X/2))/\operatorname{ad} X\}|^{-1/2}\phi(X)$$

is a function on an invariant neighbourhood of 1 in the Lie group. The dual map $T \to \tau_T$ takes invariant distributions in G to invariant distributions on U, and in order for the reduction to function it must be shown that it takes eigendistributions of z in the centre of the universal enveloping algebra to eigendistributions of the operator $\partial(p_z)$, where p_z is an invariant polynomial on the Lie algebra associated to z. For the Casimir operator Harish-Chandra knew this, but that was not sufficient evidence that it was true in general. So in Paris, contrary to custom, for he seldom considered special cases, he passed to the group SL(3), for which there is a second generator of the centre of degree 3, and by explicit calculation showed that $\partial(p_z)\tau_T = \lambda\tau_T$ if $zT = \lambda T$. This appears to have given him the necessary confidence that it was true in general.

The Institute for Advanced Study

In 1961-62 Harish-Chandra spent another year at the Institute for Advanced Study, which was contemplating appointing him a professor. The offer was finally made in 1963 and he returned, remaining there with only brief absences until his death.

Although his health had been undermined by overwork he had regained his vigour and was at the height of his career, about to establish the existence and the properties of the discrete series. A tall, exceedingly handsome man, already somewhat on the thin side, he was a little timid and reserved, and his intercourse was marked by a formal courtesy which did not conceal the intensity of his feelings and was often broken by a laugh or a smile, although in later years he was inclined to withdraw behind it. However in 1963 his mathematical horizons were still expanding and he was, if not easy, certainly confident and gregarious, moved by those about him and willing to gossip, his comments lacking neither the appropriate malice nor the necessary insight. He enjoyed conversation on general topics and on mathematics, where his preferred style, reflecting his enthusiasm and the pace of his work, was the monologue, but there was nothing inchoate about his thought and he spoke clearly and fluently. For those who were willing to forget their own preoccupations for an hour or two it was a great pleasure to see his ideas still red-hot from the forge.

He could listen too, respond to comments and reflect on questions, but it was difficult to turn him to topics divorced from his own concerns or to make him consider a view different from his own. However, he could change his mind and was deeply attached to younger mathematicians whose work impinged on his own and who he felt had contributed decisive ideas, even when, as sometimes happened, his initial impression had been unfavourable. The influence of R. Howe, a mathematician of a cast of mind quite different from his own whose ideas were a key to harmonic analysis on p-adic groups, is patent, and was enthusiastically acknowledged. His response to my own work, perhaps because it underlined the value of his own contributions for the theory of automorphic forms, was generous in the extreme. In later years he became convinced of the importance of J. Arthur's work on the trace formula. It and the work of L. Clozel on Howe's conjectures, to which he attached great importance, were two topics that preoccupied him at his death. He did not assimilate the ideas of others easily, but if he felt they would be useful to him he was unstinting in his efforts to put them in a form he could understand.

He was inclined to brood on his own work as well. It was a sustained, cumulative effort, and he liked to formulate explicitly the ideas that guided him. The philosophy of cusp forms, which he borrowed from the analytic theory of automorphic forms, and in terms of which he eventually cast his theory of harmonic analysis on real groups, was a favourite principle. He later transferred it to p-adic groups and introduced it as well, in a short but influential paper (1970b), into the study of representations of finite Chevalley groups. He would have regarded this as an application of the Lefschetz principle, which for him meant that real groups, p-adic groups and automorphic forms (corresponding to archimedean and non-archimedean local fields and to number fields) should be placed on an equal footing, and that ideas and results from one of these three categories should transfer to the other two. The name of the principle is one of the few traces of Harish-Chandra's early reading in algebraic geometry. In his hands it led to substantial advances in harmonic analysis on p-adic groups, but it also encouraged him to ignore the arithmetical aspects of p-adic groups and automorphic forms, which appear to be richer than the analytic.

His preferred method of proof was induction, which was particularly suited to real groups, for which he was able to reduce many problems down to SL(2, $\mathbb{R}$). He compared it to high finance. 'If you don't borrow enough you have cash flow problems. If you borrow too much you can't pay the interest.' Just the day before his death he maintained in a large gathering that 'In mathematics we agree that clear thinking is very important, but fuzzy thinking is just as important as clear thinking.' None the less, he himself, although he could be wrongheaded, was never fuzzy.

At the Institute for Advanced Study there are few formal duties, but Harish-Chandra loved to lecture on work in progress. Most years found him delivering a series of talks. Once or twice his enthusiasm encouraged him to precipitance. His first lectures on the discrete series in 1961 were abruptly broken off, to be resumed two years later when the hole was patched.

In 1968 he was named I.B.M. von Neumann Professor of Mathematics at the Institute. He was elected a Fellow of the Royal Society in 1973 and later of other academies. Such honours pleased him, but for a mathematician of his stature he received very few. He was considered for the Fields Medal in 1958, but a forceful member of the selection committee in whose eyes Thom was a Bourbakist was determined not to have two. So Harish-Chandra, whom he also placed in the Bourbaki camp, was set aside. Harish-Chandra would have been as astonished as we are to see himself lumped with Thom and accused of being tarred with the Bourbaki brush, but whether he would have been so amused is doubtful, for it had not been easy for him to maintain confidence in his own very different mathematical style in face of the overwhelming popular success of the French school in the 1950s.

He travelled little in later life, two short stays in Paris and a brief visit to India. Even vacations became rare. However, he did travel to the International Congress of Mathematicians in Moscow in 1966, at which he delivered one of the general lectures, and was delighted and flattered by I. M. Gelfand's hospitality.

In 1969 he had his first heart attack, and from then on his health was a serious concern. His physician prescribed regular exercise and Harish-Chandra complied, walking in the late afternoon in the streets near the Institute with long, rapid strides at a faster pace than many of the joggers. But he could not rest on his accomplishments and did not cease working. There was a competitive streak in him that he never recognized and never mastered. It did not let him rest. Sometimes he would press himself too hard, as in his attack on the spectral theory of Whittaker functions which yielded only on the second assault, the first too sanguine attempt having been unsuccessful. During the ensuing period of enforced rest a youthful almost ebullient Harish-Chandra reappeared, chatting easily about trivial matters and discoursing passionately about his favourite painters.

His heart grew worse and in 1982 he had a third attack, from which he never properly recovered. His last year was troubled by increasing frailty, the effects of medication and the knowledge that he had little time left. A conference to celebrate his 60th birthday was planned for April 1984 but he was not to live to participate in it. A similar conference in honour of Armand Borel was held in Princeton in October 1983, and was attended, the fields of the two being so close, by many friends and colleagues of Harish-Chandra. No one knows why or how, but for the week of the conference, his vigour and force reasserted themselves. Princeton's warm, clear autumn weather prevailed and between lectures at the conference, on a lawn or a terrace of the Institute, he was the centre of a lively crowd, expressing his views on a variety of topics. On Sunday 16 October, the last day of the conference he and Lily had many of the participants to their home. He was a sparkling host. In the late afternoon, after the guests had departed, he went for his customary walk, and never returned alive. His ashes were spread in Princeton and immersed in the Ganges at Allahabad.

Mathematical work

Harish-Chandra's *Collected works* were published in 1984 and contain essays by V. S. Varadarajan, N. Wallach and R. Howe that provide a comprehensive survey of Harish-Chandra's mathematical papers, describing not only his general theory and specific contributions but also the context in which they were produced. So the following description of his achievements will be brief. His papers are with few exceptions cumulative, and to some extent accretive. It appears that by the early 1950s he had already glimpsed the outlines of the theory of harmonic analysis on real semisimple groups, and in the next ten years he marched towards it with formidable determination and resourcefulness, inventing techniques and constructions as he advanced. Even after the wave of advance had crested in the discrete series and its force been partly diverted into other channels, the tenacity in the search for solutions to technical difficulties which was a characteristic of Harish-Chandra's style remained. It helps when reading his papers if one can isolate the places where severe and steady pressure has had to be applied and separate them from the stretches where experience and strength sufficed. I have attempted something of the sort but my limited familiarity with many of the papers does not permit much confidence. With time some of Harish-Chandra's arguments have been simplified and some specific results have been shown to be consequences of other general theories. I have not alluded to any of this or to subsequent developments, nor to papers he may have left in manuscript form. They have yet to be examined.

Apprenticeship

In a six-month period in 1948, beginning about half a year after his arrival in Princeton, Harish-Chandra wrote five papers giving new proofs or extensions of existing theorems in the theory of Lie algebras and, to some extent, groups. The influence of Chevalley on these early papers is manifest. Among other things Ado's theorem affirming the existence of faithful representations of a Lie algebra over a field of characteristic zero is proved and generalized. So is the Tannaka duality theorem, for Lie algebras and for groups.

The paper (1951a) written in 1950 is transitional. The first part, in which the remarkable ability to deal with the abstract semisimple Lie algebra that was a hallmark of Harish-Chandra is already highly developed, provides the first general proof of the existence of the semisimple Lie algebra attached to a Cartan matrix. He establishes it at the same time as he proves the existence by purely algebraic means of a finite-dimensional irreducible representation of the algebra with a given highest weight. This is of course the basic theorem of the subject, and had been proved before, with quite different methods, by Cartan and by Weyl. Harish-Chandra attributes some of the ideas in his construction to Chevalley.

In the remaining three sections of this long paper he strikes out on his own. He considers infinite-dimensional representations and initiates the theory of $(\mathfrak{G}, K)$-modules, but only for complex semisimple Lie algebras, showing in effect that there are only finitely many irreducible representations with a given infinitesimal character and containing a given K-type, and that a given K-type occurs only a finite number of times in a given irreducible representation. In addition he introduces for any semisimple Lie algebra the isomorphism from the centre of the universal enveloping algebra to the algebra of elements invariant under the Weyl group in the symmetric algebra of a Cartan subalgebra. It is now known as the Harish-Chandra isomorphism.

Following I. M. Gelfand and M. A. Naimark, but working with a general complex semisimple group, he introduces in the last section of the paper the unitary principal series.

Foundations of infinite-dimensional representation theory

These were created rapidly, so that by 1954 he had already turned to purely analytic problems: harmonic analysis and the existence of the discrete series.

The main technical achievements were the existence of analytic vectors, which allows one to purge the theory of inessential functional-analytic features and thus to pass to the almost purely algebraic $(\mathfrak{G}, K)$-modules; and the subquotient theorem, from which one can deduce the existence of the distribution character as well as an integral formula for the matrix coefficients of an irreducible representation which Harish-Chandra, under the influence of the theory of automorphic forms, later called the Eisenstein integral.

Grappling with the Plancherel formula

The distribution character $f \to T_\omega(f)$ associated to an equivalence class of irreducible unitary representations once introduced, the Plancherel formula is a formula,

$$f(1) = \int_{\mathcal{E}} T_\omega(f)\, d\omega,$$

valid for smooth compactly supported functions on the group. The integration is to be taken over an explicitly described collection of inequivalent irreducible representation of the group with respect to an explicit measure . For complex classical groups such a formula was found by Gelfand and Naimark. To prove it one combines integration formulas on the group resulting from the circumstance that every element lies in a Borel subgroup with elementary Fourier analysis. Harish-Chandra recognized (1951f, 1954c) that the proof could be extended to an arbitrary complex semisimple group but not to a real semisimple group with more than one conjugacy class of Cartan subgroups.

He began to attack the problem for real groups on several fronts. He proved the Plancherel formula for SL(2) by explicit, elementary calculations (1952), using the existence of the discrete series, and understood that as far as the representations needed for the Plancherel theorem were concerned the critical point was the construction of the square-integrable representations, often called the discrete series. The notion of a square-integrable representation had also been extracted from the results of Bargmann by Godement, but what is striking is that Harish-Chandra recognized, so far as I know before bounded symmetric domains and automorphic forms became popular topics, in the work of Bargmann and that of Gelfand and Graev on $\mathrm{SL}(n, \mathbb{R})$ the technique of constructing square-integrable representations on the L^2-sections of holomorphic vector bundles. He also showed, although he expressed himself differently and the significance of the fact was not to be realized until much later when, after his proof of the existence of the full discrete series, explicit constructions were sought, that (in current terminology) only the holomorphic discrete series could be realized on cohomology groups in degree 0.

He was marshalling other techniques as well, bringing the spectral theory of ordinary differential equations and Fourier analysis to bear. He also observed that the character of an irreducible representation was an eigendistribution of the centre of the universal enveloping algebra. This allowed him to show (the argument is not difficult) that the character is, at least on the regular set, a function given by a quotient of a rather simple form (1955b, 1956c). There is a well-determined denominator, and a numerator which is a linear combination of elementary functions. It is the coefficients of the numerator that have to be determined in later uniqueness arguments. It is by no means clear that the character itself is a function on the singular set as well. In particular the denominator is 0 there. So one cannot say exactly when Harish-Chandra began to suspect that the quotient was everywhere locally integrable and represented the distribution, but it was certainly not long after 1955.

The deepest sequence of papers from this period is perhaps that devoted to the limit formula, which expresses the value $f(0)$ of a smooth, compactly supported function f on the Lie algebra as a limit of derivatives of its orbital integrals. One supposes that he hoped to apply the results to the Plancherel formula itself and that he was at the time unaware how important they would be for the construction of the discrete series. Here, as everywhere in the work of Harish-Chandra, there are basic identities for differential operators which result from suppressing, for various reasons, coordinates that are in some sense polar and keeping only radial coordinates. The identity that expresses the orbital integrals of zf in terms of those of f when z is an invariant differential operator with constant coefficients on the Lie algebra is, along with a similar identity on the group, the critical one in these papers and is still basic. Harish-Chandra writes it as $\phi_{\partial(p)f} = \partial(\bar{p})\phi_f$. Another technique that appears for the first time here is reduction to the semiregular elements. It was to be used over and over again. Otherwise the limit formula results from combining integration formulas with elementary techniques from Fourier analysis, and from properties of the fundamental solution of some second-order hyperbolic equations with constant coefficients.

The search for an explicit Plancherel formula is a problem in spectral theory. The formula can also be regarded as expressing a function transforming on the left and right under given irreducible representations of a maximal compact subgroup K as an integral of matrix coefficients of irreducible representations. Apparently attempting to obtain a handle on the explicit measure in the Plancherel formula Harish-Chandra considered in reference 1958a,b functions bi-invariant under K, for which he seems to have (correctly) believed that the discrete series was irrelevant, so that he had all the ingredients of their harmonic analysis at his disposal.

Here one is dealing with a higher-dimensional version of the classical spectral theory on a half-line. The elementary spherical functions which are the elements of the expansion satisfy differential equations that, when the K-invariance is taken into account, yield an overdetermined system. The central topic of the papers is the asymptotic behaviour of the spherical functions, for Harish-Chandra shows that the Plancherel measure is given, as in the classical theory, by the coefficients of the asymptotic expansion. He shows that it has an asymptotic expansion by generating a series by recursion, checking convergence and then verifying that it satisfies the necessary differential equations by interpolating between those values of the parameter that correspond to finite-dimensional representations. For the finer study of the expansion he is able to reduce the equations to a form that enables him to exploit along rays a method much like variations of parameters. The coefficient that appeared in the asymptotic expansion he labelled the c-function. He gives an integral formula for it and proves what he later referred to in a more general context as the Maass-Selberg relations. The basic ingredients of his harmonic analysis, especially the weak inequality and the Schwartz space, are implicit in these papers. All that is missing, apart of course from the discrete series, is the Bhanu–Murty–Gindikin–Karpelevich device for reducing the calculation of the c-function to the rank-one case. However, the explicit formula for rank-one groups does appear.

The discrete series

The basic results were announced in his papers of 1963. The first is that every invariant eigendistribution, in particular every character, is a locally summable function. The others, more difficult to state, provide the full discrete series. However, only later was he able to describe explicitly the characters of the representations of the discrete series on the elliptic set and thereby parametrize them. These are the central results in the representation theory of real semisimple groups and the proofs were long and arduous, requiring several difficult technical steps which were dealt with in a sequence of papers culminating in papers 1965c and 1966b, now referred to as *Discrete series* I and II.

There are two forms to the theorem that an invariant eigendistribution is a function. On the Lie algebra the differential operators of which the distribution is to be an eigendistribution are the invariant constant coefficient differential operators. On the group they are the elements of the centre of the universal enveloping algebra. The theorem is first proved for the Lie algebra and then transferred to the group.

Harish-Chandra first proves, in a local form, that an invariant eigendistribution of the Casimir operator that is supported on the nilpotent elements is zero.

The crux of the matter is, of course, to show that there are transverse directions on the symbol of the differential operator that there is no possibility of cancelling. The Jacobson–Morosov lemma, which was to reappear in his work on p-adic groups, is an important tool.

Local summability on the Lie algebra is proved in reference 1965a. As already remarked it is not difficult to show that on the regular set the distribution is given by a function F. Moreover he has proved in a preceding paper (the key being the identity $\phi_{\partial(p)f} = \partial(\bar{p})\phi_f$ for orbital integrals) that F is locally summable and thus defines a distribution T_F. So he needs to show that $T - T_F = 0$. By induction on the dimension of the algebra he shows that it is supported on the nilpotent elements. For some r and some c the equation $(\partial(\omega) - c)^r T = 0$ is satisfied, $\partial(\omega)$ being the Casimir operator. By an induction on r he is allowed to assume $r = 1$. Then $(\partial(\omega) - c)(T - T_F) = T_{\partial(\omega)F} - \partial(\omega)T_F$. Since the right side of this equation is a distribution defined b y a function it can be studied by integration by parts. Examining it around semiregular elements where it is shown to be 0 by induction and by reduction to $\mathrm{SL}(2, \mathbb{R})$, which can be treated directly, one shows that it is 0. Thus $(\partial(\omega) - c)(T - T_F) = 0$. Since $T - T_F$ is supported on the set of nilpotent elements it follows from the first paper of the sequence that it is 0.

In reference 1965a he also proves the theorem that is critical for transferring the results from the algebra to the group. It states that an invariant differential operator that annihilates all invariant functions also annihilates all invariant distributions.

The existence and uniqueness of the invariant eigendistributions, which turn out to be the discrete series characters, is first proved on the Lie algebra. There the existence is in essence proved by writing the distributions down as an explicit Fourier transform. The existence on the group is proved by transferring in patches invariant distributions from the algebra to the group by the duals of transformations on functions of the form $f \to \phi$ with $\phi(X) = \xi(X)f(\exp X)$, ξ being a fixed function. The distributions are specified by their restriction to the open set of regular elliptic elements. Uniqueness is obtained by moving across semiregular elements to the open sets determined by the regular elements in other conjugacy classes of Cartan subgroups, using the growth conditions imposed to force most of the constants in the numerators to be 0. It involves also integration by parts and the use of the differential equations to match the constants across semiregular elements.

In 1966 he finished his study of the discrete series (1966a,b), proving that the eigendistributions he had constructed are indeed characters of square-integrable representations, and turned to the harmonic analysis. The function Ξ, which defines the rate of decay demanded of functions in the Schwartz space $\mathscr{C}(G)$, appears and half of reference 1966b is devoted to the basic properties of this space. To show that the eigendistributions that he has constructed and labelled θ_λ are characters of square-integrable representations he must, first of all, verify that their Fourier coefficients with respect to a maximal compact subgroup are square integrable, or better, lie in $\mathscr{C}(G)$. The eigendistributions are tempered, in other words they lie in the dual of $\mathscr{C}(G)$, and thus by the theory of $\mathscr{C}(G)$ so are their Fourier coefficients, which therefore satisfy the weak inequality, the rate of growth permitted to K-finite functions in the dual of $\mathscr{C}(G)$. However, the differential equations satisfied by θ_λ are passed on, although in somewhat altered form, to its Fourier coefficients, and they imply that slow growth must be rapid decay.

To show that the eigendistributions are tempered is a serious matter. The argument is convoluted and is made to rely ultimately on the maximum principle for the Laplace-Beltrami operator on G, although that could be avoided. Because one of the invariant distributions, later called the Steinberg character by Harish-Chandra, is constant on the set G_B of conjugates of regular elliptic elements, the necessary estimates can be reduced to one for $\int_K \phi_B(xk)\,dk$ if ϕ_B is the characteristic function of G_B, provided that the Fourier coefficient of the Steinberg character with respect to the trivial representation of K is 0. It is in fact 0 for all the eigendistributions. This is because an elementary estimate forces it to go to zero at infinity while the differential equations force it to grow.

The final major task (1966b) was to prove that the invariant eigendistributions he had constructed were up to sign the characters of the discrete series representations. The argument is similar to that of Weyl for compact groups and employs the orthogonality relations whose proofs are based on ideas from one of Harish-Chandra's first papers (1956b) on square-integrable representations.

Harmonic analysis

By the time he had completed the papers on the discrete series Harish-Chandra had all the techniques necessary for the development of the harmonic analysis at his disposal. However, in 1966 he gave a series of expository lectures on Eisenstein series and the analytic theory of automorphic forms, and these strongly influenced his view of harmonic analysis on a semisimple group and his presentation of it. In the lectures (1970a,c) the harmonic analysis of the space $L^2(G)$ is cast in the mould of that of $L^2(\Gamma\backslash G)$. Cusp forms appear, as do Eisenstein integrals, and cuspidal parabolic subgroups are displacing Cartan subgroups. The Schwartz space is there from before and from now on the harmonic analysis would be couched in terms of it and not of $L^2(G)$. As for ordinary Fourier analysis this permits the formulation of more precise theorems. He introduces the space $\mathscr{C}_i(G)$ attached to the ith associate class of parabolic subgroups and defined by means of the constant terms

$$f^P(g) = \int_N f(ng)\,dn$$

of functions in $\mathscr{C}(G)$ and verifies the direct sum decomposition

$$\mathscr{C}(G) = \oplus_i \mathscr{C}_i(G).$$

In addition he introduces the spaces $\mathfrak{A}(G, \tau)$, which are analogues of spaces appearing in the theory of automorphic forms, and for a function f in one of them defines, with the help of the differential equations it satisfies, the weak constant term f_P. The Eisenstein integral yields functions in the spaces $\mathfrak{A}(G, \tau)$. He is still having trouble with the analytic continuation and functional equations, which he wants to put in the form familiar from the theory of Eisenstein series. They would be dealt with in the lecture (1972) in which the Maass-Selberg relations suggested by the analytic theory of automorphic forms also occur. The Plancherel formula is proved, although the measure is not given as explicitly as in reference 1972.

The proofs of these results appeared in three long papers (1975, 1976a,b). (The techniques of the first two papers have their origins in the two early papers on spherical functions (1958a,b)). The existence and properties of the weak constant term that allow him to prove that wave packets lie in $\mathscr{C}(G)$ are proved with the help of the differential equations by variants of the method of variation of parameters. There is also a critical formula that yields the constant term of a wave packet as an integral of the weak constant term of its elements. The proof is subtle and appears, in simpler form, already in reference 1958b. There one has a double integral

$$\int d\bar{n} \int a(\lambda)\phi_\lambda(\bar{n}h)\,d\lambda$$

to evaluate, the ϕ_λ being elementary spherical functions. Harish-Chandra replaces h by $h \exp tH$, uses the differential equations to show that this does not affect the value of the integral, and then lets t approach infinity, replacing ϕ_λ by its asymptotic expansion.

Although the Maass–Selberg relations show that all members of the weak constant term of an Eisenstein integral have the same weight on the unitary axis, in the domain where the real part of the parameter lies in the positive chamber one member dominates because of its exponent, and can be evaluated to yield a relation between the c-function and the intertwining operators. For spherical functions the argument appears in reference 1958b, and just as for spherical functions, the Bhanu–Murty–Gindikin–Karpelevich technique then reduces the calculation of the Plancherel measure to the case of maximal cuspidal parabolic subgroups.

The proof of the Plancherel formula (1976b) uses the limit formula (1957e) but perhaps not in the way suggested by paper 1970c. Using a measure $\mu(w, v)$ defined in terms of the c-function, or the intertwining operators, he builds wave packets. Then, using the limit formula, the explicit formula for the characters of the discrete series on the elliptic elements, and the expressions for the constant terms of the wave packets in terms of the c-function, he is able to evaluate their inner products with Eisenstein integrals. After that the Plancherel formula for K-finite functions follows from formal principles, provided that one can control the growth of $\mu(w, v)$ for large values of the parameters. For this he uses an explicit expression for it that he can obtain from the relative-rank one case by the product formula. For relative-rank one the alternative approach to the Plancherel formula (1970c), which exploits the limit formula (1957e), is manageable. So he applies it to evaluate a function which must be μ.

Discrete and finite groups

In the 1950s a large number of mathematicians began to consider the notion of an automorphic form for discrete subgroups of arbitrary semisimple groups. The domain attracted Harish-Chandra and he made two contributions. The analytic theory of automorphic forms cannot begin until one knows that the space of automorphic forms defined by a given discrete group, a given representation of a maximal compact subgroup, and a given ideal in the centre of the universal enveloping algebra, is finite-dimensional. The first general theorem of this type is proved in reference 1959a. The argument, based on Godement, anticipates in many respects that of the definitive results obtained a few years later, whose proof uses the general reduction theory of the paper (1962) written in collaboration with Borel. This reduction theory, which issued from the classical reduction theory and subsumes it, has been incorporated into the very foundations of the theory of automorphic forms. It yields, in particular, the finiteness of the volume of the fundamental domain for an arbitrary arithmetical subgroup of a semisimple group and the criterion, conjectured by Godement, for its compactness.

He also introduced the notion (1970b) of a cusp form into the representation theory of groups over finite fields. The subsequent theory has been erected upon the framework it provides.

Groups over p-adic fields

The representation theory of reductive groups over non-archimedean fields was a preoccupation of Harish-Chandra from the late 1960s. His goal was to carry the harmonic analysis of groups over $\mathbb{R}$ to groups over p-adic fields. The notes (1970d) are his first contribution to the subject, and he plunges in with the proof that an irreducible square-integrable representation is admissible and therefore possesses a character. It was perhaps the discovery of this proof that convinced him that a theory for p-adic groups parallel to the real theory could be developed. Oddly enough, in Harish-Chandra's theory, the general proof that an irreducible unitary representation is admissible was to come only at the end and only in characteristic 0, although in the meantime J. Bernstein had, by other means, proved it in general.

Two results from reference 1970d which was in some respects provisional, would be incorporated in reference 1978, whose central result is that the character of an irreducible admissible representation is given by a locally summable function. For real groups the asymptotic behaviour of the orbital integrals near the singular points is analysed with the help of differential equations. For p-adic groups the asymptotic behaviour is described by fewer elements, the Shalika germs, but they are more difficult to get a handle on. In a long chapter (1970d) Harish-Chandra succeeded in establishing, with the help of the Jacobson–Morosov lemma, important homogeneity properties of these germs, which yielded estimates from which he could establish the local summability of a function majorizing characters. The technique for estimating supercuspidal characters by expressing them as orbital integrals of matrix coefficients that has its origins in reference 1956b is developed in reference 1970b and appears in reference 1978 for the invariant distributions defined by cusp forms on the Lie algebra.

As for real groups the tactic (1978) is to work first on the Lie algebra and then to pass to the group. For p-adic Lie algebras there is no strict notion of invariant eigendistribution, but there is a substitute, the Fourier transforms $\hat{T}$ of invariant distributions T supported on orbits. Since the Fourier transforms of functions supported on a small compact set are uniformly locally constant one may for the local theory even allow the distribution T to have support in a thickening of the orbits. This is made precise by a finiteness theorem of Howe, which then becomes the key to the theory on the Lie algebra. The passage from the algebra to the group requires the introduction of a class of invariant distributions that is large enough to include the characters of irreducible admissible representations, and to permit localization and reduction to the centralizers of semisimple elements. Once again the key notion, that of (G, K)-admissibility, is extracted from results of Howe.

The notion appears already in his paper of 1973, which like reference 1977b is a summary of results. The complete theory was expounded by A. Silberger in his *Introduction to harmonic analysis on reductive p-adic groups*. For functions, like matrix coefficients, that are K-finite on both sides, the Hecke algebra provides an adequate substitute for the differential operators. However, Harish-Chandra (1970b) was unable to utilize the condition of Hecke-finiteness as he had used the differential equations to study the asymptotic behaviour of functions in the space $\mathfrak{A}(G,\tau)$. The lock was turned by Jacquet with his introduction of the module now named after him and the path opened to the development (1973) of the elements of harmonic analysis: asymptotic expansions, the Schwartz space, wave packets and the Plancherel measure. It remained to prove that the wave packets exhaust the Schwartz space. For a semisimple group this amounts to two closely related theorems: the trivial representation of a given open compact subgroup K is contained in only finitely many discrete series representations, and a K-finite cusp form which satisfies the weak inequality is Hecke-finite. These are proved in reference 1977b, or rather deduced as consequences of another statement whose proof is not given. Harish-Chandra has stated that it was inspired by Arthur's integral formula for the character of a square-integrable representation of a real group.

Honours

Harish-Chandra was a Guggenheim Fellow in 1957-58 and a Sloan Fellow from 1961 to 1963. He was elected a Fellow of the Royal Society in 1973. He was elected Fellow of the Indian Academy of Sciences and of the Indian National Science Academy in 1975 and of the National Academy of Sciences of the U.S.A. in 1981. He was an Honorary Fellow of the Tata Institute of Fundamental Research, Bombay. He was awarded honorary degrees by Delhi University in 1973 and Yale University in 1981. He received the Cole Prize of the American Mathematical Society in 1954 and the Srinivasa Ramanujan Medal of the Indian National Science Academy in 1974.

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