Code : **M01**

Roll Number: ☐☐☐☐ ☐☐☐☐ ☐☐☐☐☐☐ **Time: 3 Hours**

Number of Questions: 30 **Max Marks: 100**

INSTRUCTIONS

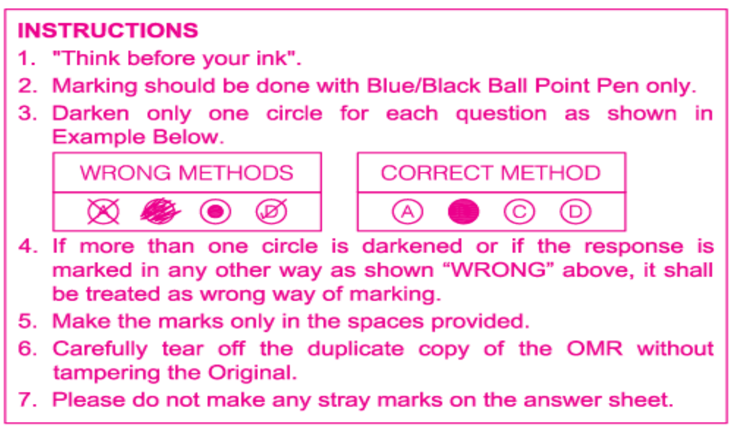
1. Use of mobile phones, smartphones, ipads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

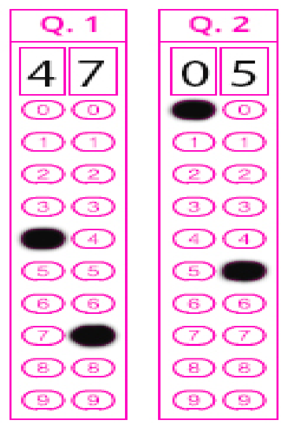
2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen**. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer. .

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.

4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.





6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.

7. Questions 1 to 8 carry 2 marks each; questions 9 to 21 carry 3 marks each; questions 22 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

1. Let $ABCD$ be a trapezium in which $AB \parallel CD$ and $AB = 3CD$. Let E be the midpoint of the diagonal BD. If $[ABCD] = n \times [CDE]$, what is the value of n? (Here $[\Gamma]$ denotes the area of the geometrical figure Γ.)

2. A number N in base 10, is 503 in base b and 305 in base $b+2$. What is the product of the digits of N?

3. If $\sum_{k=1}^{N} \frac{2k+1}{(k^2+k)^2} = 0.9999$ then determine the value of N.

4. Let $ABCD$ be a rectangle in which $AB + BC + CD = 20$ and $AE = 9$ where E is the mid-point of the side BC. Find the area of the rectangle.

5. Find the number of integer solutions to $\Big||x| - 2020\Big| < 5$.

6. What is the least positive integer by which $2^5 \cdot 3^6 \cdot 4^3 \cdot 5^3 \cdot 6^7$ should be multiplied so that, the product is a perfect square ?

7. Let ABC be a triangle with $AB = AC$. Let D be a point on the segment BC such that $BD = 48\frac{1}{61}$ and $DC = 61$. Let E be a point on AD such that CE is perpendicular to AD and $DE = 11$. Find AE.

8. A 5-digit number (in base 10) has digits k, $k+1$, $k+2$, $3k$, $k+3$ in that order, from left to right. If this number is m^2 for some natural number m, find the sum of the digits of m.

SPACE FOR ROUGH WORK

9. Let ABC be a triangle with $AB = 5$, $AC = 4$, $BC = 6$. The internal angle bisector of C intersects the side AB at D. Points M and N are taken on sides BC and AC, respectively, such that $DM \parallel AC$ and $DN \parallel BC$. If $(MN)^2 = \frac{p}{q}$ where p and q are relatively prime positive integers then what is the sum of the digits of $|p - q|$?

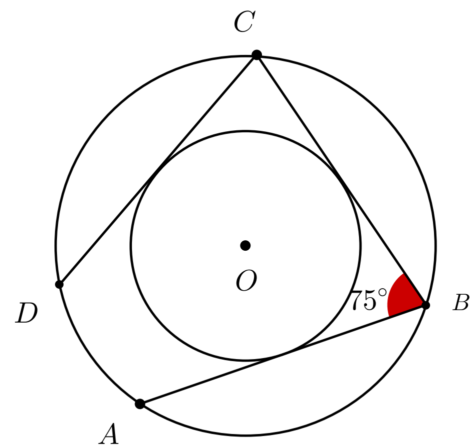
10. Five students take a test on which any integer score from 0 to 100 inclusive is possible. What is the largest possible difference between the median and the mean of the scores? (The median of a set of scores is the middlemost score when the data is arranged in increasing order. It is exactly the middle score when there are an odd number of scores and it is the average of the two middle scores when there are an even number of scores.)

11. Let $X = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ and

$$S = \{(a, b) \in X \times X : x^2 + ax + b \text{ and } x^3 + bx + a \text{ have at least a common real zero}\}.$$

How many elements are there in S?

12. Given a pair of concentric circles, chords $AB, BC, CD, \ldots$ of the outer circle are drawn such that they all touch the inner circle. If $\angle ABC = 75°$, how many chords can be drawn before returning to the starting point?



SPACE FOR ROUGH WORK

13. Find the sum of all positive integers n for which $|2^n + 5^n - 65|$ is a perfect square.

14. The product $55 \times 60 \times 65$ is written as the product of five distinct positive integers. What is the least possible value of the largest of these integers?

15. Three couples sit for a photograph in 2 rows of three people each such that no couple is sitting in the same row next to each other or in the same column one behind the other. How many arrangements are possible?

16. The sides x and y of a scalene triangle satisfy $x + \frac{2\Delta}{x} = y + \frac{2\Delta}{y}$, where Δ is the area of the triangle. If $x = 60, y = 63$, what is the length of the largest side of the triangle?

SPACE FOR ROUGH WORK

17. How many two digit numbers have exactly 4 positive factors? (Here 1 and the number n are also considered as factors of n.)

18. If

$$\sum_{k=1}^{40}\left(\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}\right)=a+\frac{b}{c}$$

where $a, b, c \in \mathbb{N}, b < c, \gcd(b, c) = 1$, then what is the value of $a + b$?

19. Let $ABCD$ be a parallelogram . Let E and F be midpoints of AB and BC respectively. The lines EC and FD intersect in P and form four triangles APB, BPC, CPD and DPA. If the area of the parallelogram is 100 sq. units, what is the maximum area in sq. units of a triangle among these four triangles?

20. A group of women working together at the same rate can build a wall in 45 hours. When the work started, all the women did not start working together. They joined the work over a period of time, one by one, at equal intervals. Once at work, each one stayed till the work was complete. If the first woman worked 5 times as many hours as the last woman, for how many hours did the first woman work?

21. A total fixed amount of N thousand rupees is given to three persons A, B, C, every year, each being given an amount proportional to her age. In the first year, A got half the total amount. When the sixth payment was made, A got six-seventh of the amount that she had in the first year; B got Rs 1000 less than that she had in the first year; and C got twice of that she had in the first year. Find N.

SPACE FOR ROUGH WORK

22. In triangle ABC, let P and R be the feet of the perpendiculars from A onto the external and internal bisectors of $\angle ABC$, respectively; and let Q and S be the feet of the perpendiculars from A onto the internal and external bisectors of $\angle ACB$, respectively. If $PQ = 7$, $QR = 6$ and $RS = 8$, what is the area of triangle ABC?

23. The incircle Γ of a scalene triangle ABC touches BC at D, CA at E and AB at F. Let r_A be the radius of the circle inside ABC which is tangent to Γ and the sides AB and AC. Define r_B and r_C similarly. If $r_A = 16$, $r_B = 25$ and $r_C = 36$, determine the radius of Γ.

24. A light source at the point $(0, 16)$ in the coordinate plane casts light in all directions. A disc (a circle along with its interior) of radius 2 with center at $(6, 10)$ casts a shadow on the X axis. The length of the shadow can be written in the form $m\sqrt{n}$ where m, n are positive integers and n is square-free. Find $m + n$.

25. For a positive integer n, let $\langle n \rangle$ denote the perfect square integer closest to n. For example, $\langle 74 \rangle = 81, \langle 18 \rangle = 16$. If N is the smallest positive integer such that

$$\langle 91 \rangle \cdot \langle 120 \rangle \cdot \langle 143 \rangle \cdot \langle 180 \rangle \cdot \langle N \rangle = 91 \cdot 120 \cdot 143 \cdot 180 \cdot N$$

find the sum of the squares of the digits of N.

SPACE FOR ROUGH WORK

26. In the figure below, 4 of the 6 disks are to be colored black and 2 are to be colored white. Two colorings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same.

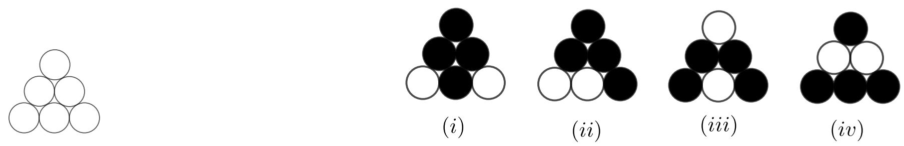


Fig 1.

There are only four such colorings for the given two colors, as shown in Figure 1. In how many ways can we color the 6 disks such that 2 are colored black, 2 are colored white, 2 are colored blue with the given identification condition?

27. A bug travels in the coordinate plane moving only along the lines that are parallel to the x axis or y axis. Let $A = (-3, 2)$ and $B(3, -2)$. Consider all possible paths of the bug from A to B of length at most 14. How many points with integer coordinates lie on at least one of these paths?

28. A natural number n is said to be *good* if n is the sum of r consecutive positive integers, for some $r \geq 2$. Find the number of good numbers in the set $\{1, 2, \ldots, 100\}$.

29. Positive integers a, b, c satisfy $\dfrac{ab}{a-b} = c$. What is the largest possible value of $a+b+c$ not exceeding 99?

30. Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number, $a+1$ divides $b-1$ and b divides a^2+a+2.

SPACE FOR ROUGH WORK

ROUGH WORK