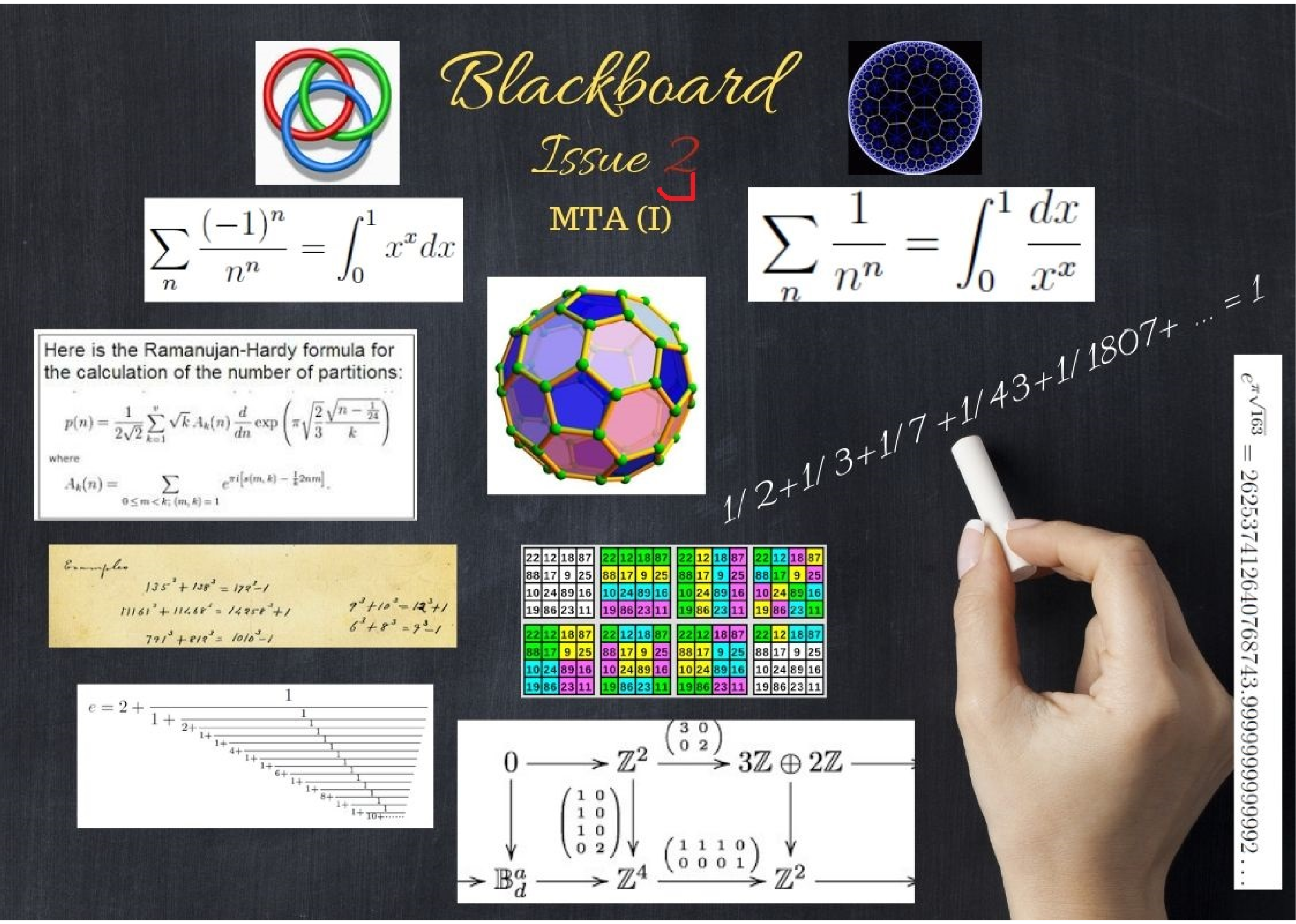
Blackboard
Issue 3
MTA (I)
$\sum_n \frac{(-1)^n}{n^n} = \int_0^1 x^x dx$
$\sum_n \frac{1}{n^n} = \int_0^1 \frac{dx}{x^x}$
Here is the Ramanujan-Hardy formula for the calculation of the number of partitions:
$p(n) = \frac{1}{2\sqrt{2}} \sum_{k=1}^{v} \sqrt{k}\, A_k(n) \frac{d}{dn} \exp\left(\pi \sqrt{\frac{2}{3}} \frac{\sqrt{n-\frac{1}{24}}}{k}\right)$
where
$A_k(n) = \sum_{0 \le m < k;\ (m,k)=1} e^{\pi i \left[s(m,k) - \frac{1}{k} 2nm\right]}$.
1/ 2+1/ 3+1/ 7 +1/ 43+1/ 1807+ ... = 1
$e^{\pi\sqrt{163}} = 262537412640768743.99999999999992\ldots$
Examples
$135^3 + 138^3 = 172^3 - 1$
$11161^3 + 11468^3 = 14258^3 + 1$
$9^3 + 10^3 = 12^3 + 1$
$6^3 + 8^3 = 9^3 - 1$
$791^3 + 812^3 = 1010^3 - 1$
$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cdots}}}}}$

# Contents

# 1 Editorial

This third issue of Blackboard, the bulletin of the MTA(I) took quite a while in coming to fruition. The crisis due to the ravaging pandemic has understandably been one of the principal reasons for the delay in production. However, despite these difficulties, there have been some advantages as well in bringing out this particular issue somewhat later than when we hoped to.

In September, the annual MTA (India) conference was held online with a strong participation of teachers from all over the country. As a part of the conference, several teachers shared their experiences with novel teaching methods they has to learn in order to teach. In the conference, there was a panel discussion on aspects of the new NEP as well. The panel discussion and some of the experiences of teachers with online teaching processes have been detailed in articles here. In a very relevant article, Tathagata Sengupta has inaugurated a penetrative discussion on 'Crisis, Nation State and Education'. It is perhaps timely and worthwhile to share these writings with teachers apart from other mathematics articles that Blackboard contains. The Press Release announcing the annual conference of the MTA (I) is also included.

As indicated above, we also have a variety of mathematical topics covered in this issue. Raghunathan talks about 'Mathematics - art that would rather be science'. Muralidharan shares a nice mathematics olympiad type of problem as a case study to demonstrate how technology could assist exploration in mathematics. Kesavan discusses a version of Brouwer's fixed point theorem and shows how it can be used in solving equations in Hilbert spaces. Amber Habib has written a very interesting article on some historical aspects of mathematics. Talking about instances from ancient Indian and Mesopotamian mathematics, he suggests how they can be useful at certain stages of school to promote geometric manipulations as precursors to algebraic ones. This would hopefully lead to a more unified view of these topics. In an extremely interesting note, Jyotirmoy Sarkar talks about experimental projects that can be done in school demonstrating the role of mathematical reasoning in exploring manifold ways to label the face of a half-cube. There are beautiful pictures as well and this topic is rich enough for teachers to not only use this very effectively in their classes to generate interest among students, but also to encourage them to explore further.

... B. Sury, Indian Statistical Institute Bangalore.

## 2 Mathematics Teachers' Association (India)

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Mumbai, 400088 INDIA
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The Mathematics Teachers' Association (India) hosted its Second Annual Conference (2020) on September 5 and 6, 2020. The conference was held online and focused on the theme of "Mathematics Education in Times of the Covid-19 Pandemic".

The Covid-19 pandemic has brought before us unprecedented challenges. Education is especially disrupted due to the various lockdown and distancing measures in place. Many teachers and students across the country are struggling to cope with the challenges while keeping teaching and learning alive.

While access to online connectivity and devices is a major issue, teachers are rising to the challenge and providing students many creative ways of learning mathematics. But ensuring engagement of students and assessment constitute a major challenge. The Conference featured discussion forums where mathematics teachers voiced the challenges, efforts, and innovations they undertook to achieve successful teaching.

In the discussion forum for elementary teachers, Sibi Sebastian, a Kendriya Vidyalaya Principal from Bijapur described how teachers had gone from being completely unfamiliar with online technologies to become reasonable experts in a short time. Neeta Batra, from the Directorate of Education, New Delhi, said that blending different tools with online teaching, such as sending recorded audios over whatsapp helped students. To reach those who could not be contacted, she made student leaders, who then contacted and brought in their classmates. Pralhad Kathole, a primary teacher from Maharashtra showed how during the pandemic he visited hamlets of tribal Maharashtra to continue education and helping these disadvantaged communities to identify knowledge sources within community. Anchal Chomal from Azim Premji University provided a framework for meaningful assessment.

In the discussion forum for secondary teachers, Ms Kalpana Kannan, headmistress of Greater Chennai Corporation School, Kotturpuram, spoke of how initially their work was mainly on accessing each student and ensuring they were getting enough to eat, since the students in her school come from the poorest sections. They got crowdfunding to get one smartphone to each student, and videos of classes and other material are sent on these smartphones. Dr. Shailesh Shirali said that "'Undue importance is given to assessment in our culture. Could we rethink the role that assessment plays in education?"

In the higher education discussion forum, Jyoti Bhola, Assistant Professor from Hansraj College, spoke about the different digital platforms that she used during this pandemic to stay connected to her students, and their positive and negative aspects. Anshu Gupta, a faculty from Ambedkar University, spoke from the point of view of applied mathematics with special reference to the mathematics/quantitative methods related to decision sciences and management education. Tathagata Sengupta, a faculty from Homi Bhabha Centre for Science Education, raised issues of technology and equity and noted the connections between new policies and the increasing investment by private players in online education.

Two special workshops on using technology in teaching were conducted for school and college teachers.

For school teachers, the emphasis was on using free and open source (FOSS) tools. Sangeeta Gulati gave the participants a hands-on experience of using GeoGebra Classroom and the digital whiteboard. Jonaki Ghosh showed the use of spreadsheets in developing tasks that elicit mathematical thinking and Sneha Titus demonstrated some attractive online resources that could be used to engage primary students in exploring concepts.

For college teachers, Dr B. Surendranth Reddy, from S. R. T. Marathwada University, focussed on different learning management systems and free software tools such as Open Broadcaster Software (OBS), Microsoft forms, OneNote, Edmodo and digital pens, that could be used for teaching mathematics effectively during the pandemic. Dr Ajit Kumar gave an introduction to SageMath, an open source Mathematical Software system which could be integrated with mathematics teaching at the college level.

Among the highlights of the conference were two keynote talks. Prof. Neena Gupta of the Indian Statistical Institute, Kolkata, a young mathematician with outstanding research achievements, gave an inspiring talk on her research and described how her interest in mathematics was kindled. She is the youngest person in Mathematical Sciences to receive the prestigious Shanti Swarup Bhatnagar Prize in 2019 at the age of 35. Prof. Po-Shen Loh of Carnegie Mellon University, USA, who is the Coach for the USA Mathematics Olympiad team, gave an inspiring talk on "Mathematics vs COVID-19". He spoke about NOVID, a new and different way to perform Contact Tracing, which allows people to protect themselves before being exposed to COVID-19, as opposed to only isolating people after exposure.

The conference also included a panel discussion on the implications of the National Education Policy (NEP 2020) from the perspective of mathematics education both in school and college/university. Panelists raised concerns that many key terms in the NEP such as “multi-disciplinary education”, “mathematical thinking”, “Indian culture/ethos” were not well defined although some of the terms were repeated many times. It was not clear if the NEP would strengthen ongoing efforts, such as the Right to Education Act, to make quality education truly universal. Many participants expressed great concern that algorithm-based problems on subtraction and division were used to mechanically stream students, as young as those in Class 3, into groups of different ability, which can have long lasting negative effects of their self-perception and performance.

The conference was enjoyed thoroughly by teachers and ended with an online “cultural programme” where teachers displayed their multi-dimensional talents.

# 3 National Education Policy and Mathematics Education: A Panel Discussion by Shweta Naik & K. Subramaniam

Homi Bhabha Centre for Science Education, Mumbai

The Government of India has recently announced the National Education Policy (NEP) 2020. The NEP addresses a range of issues including institutional structures, pedagogy, curricula, teacher training, and technology, and has the potential for a profound impact on the state of education in India. A broad spectrum of views have been expressed in the media on the NEP. The second online annual conference of Mathematics Teachers' Association (India) hosted a panel discussion to view the NEP from the perspective of mathematics education and the role of MTA(I) as a professional teachers' organization. The panel aimed at discussing what vision of mathematics education is projected in the NEP 2020 at the school and higher education levels, and what consequences one must be alert to and what role individual math teacher as well the association could play in the context of NEP.

The discussion session had four panellists, two focusing on school and two on higher education. Prof. Anita Rampal, Faculty of Education, Delhi University and Dr. Jyoti Sethi, TGT Mathematics, Sarvodaya Kanya Vidyalaya, Delhi discussed the issues pertaining to school education. Prof Amber Habib, Dean, Shiv Nadar University and Dr. Tulsi Srinivasan, Faculty member, Azim Premji University discussed policy highlights and its implication for undergraduate education. Prof. K. Subramaniam, Director, Homi Bhabha Centre for Science Education, moderated the discussion.

## Opening Remarks: NEP and Mathematics Education

Prof. Subramaniam began by mentioning that NEP-2020 is a document that is very important from the perspective of MTA(I) and merits concern and study from its members. He envisions that MTA (I), which is a young organisation at this moment, will become a critical part of such policy development processes in near future. He suggested that as math educators and part of the teachers' association, not only do we need to plan and implement meaningful initiatives that improve student access to mathematics but also need to be updated about current thinking in the field of education on the link between education and social transformation, making understanding NEP-2020 as a vital objective.

Subramaniam opened the discussion by bringing out the intricacy of policy documents in general. He suggested that policy documents are complex, and they need to be read in a context. The terms and meanings present in such documents, come from ideological discourses and have become sharpened after several revisions. He pointed out that the current NEP-2020 document has undergone major changes since its conception and even till the last day of the release there were changes in the text of the policy. He indicated that as the policy has received laudatory responses as well as criticism in the media, it is important for an association of mathematics teachers to take a stand on it. Associations like MTA (I) can have a major role in policies like the NEP, and therefore deliberation on it needs to be part of the MTA (I) discourse. Additionally, he reminded us that mathematics is crucial not just in our academic lives but also for understanding the world and to develop a sense of equity. At this moment, he prompted the panellists and the audience to think about what NEP-2020 means from the point of view of equity, and as part of the association what could be offered to develop and sustain access to education for children from all strata of the society.

He highlighted that mathematics education figures substantially in NEP-2020, and quoted a paragraph from the document as evidence.

> "4.25. It is recognized that mathematics and mathematical thinking will be very important for India's future and India's leadership role in the numerous upcoming fields and professions that will involve artificial intelligence, machine learning, and data science, etc. Thus, mathematics and computational thinking will be given increased emphasis throughout the school years, starting with the foundational stage, through a variety of innovative methods, including the regular use of puzzles and games that make mathematical thinking more enjoyable and engaging. Activities involving coding will be introduced in Middle Stage."

Subramaniam pointed out that this specific paragraph also details out ideas for various initiatives and discussion on possible contributions that MTA(I) can possibly offer.

## NEP-2020 and Existing Constitutional Mandate of RTE

Prof. Rampal is former Dean of the Faculty of Education at the University of Delhi. She chaired the committee which developed mathematics textbooks for primary grades of National Council of Education, Research and Training (NCERT), post-National Curriculum Framework (NCF), 2005.

Rampal started the discussion on NEP-2020 in relation to school mathematics education. She remarked that there seemed a push to release the policy document in a hurry like many other things under this government. She had been part of reading and commenting on the NEP-draft since 2019, and She found that the earlier versions were more elaborate and included the definitions of terms. She called attention to how historically these policy documents have shaped the educational discourse, founded visions and commitments, and how actions based on such documents could configure society and the functioning of democracy.

Reminding the audience of the fundamental struggle that this country has experienced, she said, in several countries, the voting rights were often given on the basis of education and the property held. In the Indian context, Gandhi and Ambedkar took a strong position against such stratification. She recalled Ambedkar who said, “we have denied education for the majority, for very long and now on that basis we can not deny voting rights”. And therefore Rampal suggests that we look at the constitution as a pedagogic tool that keeps informing on ways to bring about equity in society. She advised that whenever one reads a policy document one needs to connect it with the constitution and the construct of equity as conceptualised in the constitution.

Rampal pointed out that the policy does not talk about inequity and that there is no mention of Dalit or Muslim child anywhere in the document. She found that the constitutional values are pushed towards the end and given equal status to things that are as mundane as respecting the public property. In the discussion on values, the document provides a list, in which justice and equity are given at the end with various others. This she said is a “matter of concern”.

She said that the concept of a localised curriculum - which was imposed through constitution, stating that education is a concurrent issue - is rejected in the policy. Rampal pointed out that according to NEP-2020, the curriculum now will be designed by a national body, and the state will provide local flavours and desired nuances. She finds this reference to “state” as providing flavours, a bit condescending and disapproves of this approach to curriculum development.

The other worrying aspect of the document according to Rampal is the idea of a stratified system of education. The document declares that 50% of students would be in vocational education at school and college (NEP 2020, p.44). This according to her contradicts with the Right to Education (RTE) Act (2009) in addition to contradicting to the Constitution. The RTE is applied from age 3 of a child, and Rampal found no reference to it in the NEP-2020. She found, the policy's suggestion of alternatives or multiple tracks at Grades 3, 5 and 8 overriding the RTE - which assures good quality education for all. She said the promotion of multiple pathways and models of education given in the policy document is alarming, especially given the disparities and hierarchies in the Indian society. A serious warning that she put forward was how the existing policy would mandate and legitimise already existing segregation of students based on superficial testing across many schools in Delhi. Rampal finds this whole discourse as one of denial - where education will be systemically denied to disadvantaged groups.

Lastly, she listed several issues that the policy failed to mention or elaborate upon. Some of the things omitted or underplayed in the document are teacher development, RTE, basic norms for quality, and pupil teachers ratio. Some of these are referred to as inputs that will be taken through public philanthropic partnership, which from Rampal's point of view are private organisations.

Rampal proposed specific issues on which MTA (I) could respond such as:
1. Definition of what is foundational numeracy, and description of a timeline to achieve the same.
2. Details of concerns when Class 8, 9, 10 and 11 are being clubbed together for vocational courses.
3. A more detailed description of what it means to have shared subject teachers [in the context of who might not get those]
4. Description of the reasoning, issues and context behind pushing for artificial intelligence in early grades.

## Teachers' and Students' Perspectives of NEP-2020

Dr. Sethi, a teacher for the last 15 years, has dedicated her work in making learning more meaningful and a critical process for her students. She has been teaching students from disadvantaged groups. As a teacher, she always looks forward to a new policy document to see if any opportunities are given to the teacher to do her job autonomously, with the agency over her work, or they are expected to follow other corporate educators. Sethi finds the release of the document ill-timed. She said during the pandemic, where schools are hardly running and with no opportunities for peer discussion even for teachers, “meaning-making” of the document is a quite challenging task. State Council of Education, Research and Training (SCERT), Delhi organised a competition for teachers to read the policy and prepare a slide presentation, and that had helped her in reading and understanding the document. She did not find anything new in the NEP-2020 in comparison to what has been given in the aims of Delhi education model (2015), that she has been following in her school teaching. Sethi found that the policy boldly states, “foundational literacy and numeracy for children is an urgent national mission, immediate measures to be taken to attain it in the short term” (NEP 2020, p.8). She added that the teachers are now expected to achieve this foundational numeracy by class 3, in other words by end of the class 3 students should be able to add and subtract using Indian numerals. This agenda of computational learning is perplexing for her, especially now as it takes precedence over skills of mathematisation, estimation and contextualisation as suggested in NCF (2005). She thinks focusing on solely computational skills is superficial. Citing an example of a Delhi Government initiative, she explained the mission “Buniyad” that involved using an assessment tool focusing just on computation and how it created unjustifiable student categories. The assessment tool is given at the age of 6, containing two problems - one of division using the standard algorithm and other of subtraction with borrow. Based on students’ performance they are streamed into groups: Pratibha - students successful in both algorithms, Nishtha - students successful in just subtraction algorithm, and Neo Nishtha - students not able to do subtraction or division. The tool created fear of failure among students and teachers and they all responded to it by adopting rote memorisation. This assessment was prioritised over everything else - students’ vacation or any other activities. Sethi expressed great concern about such streaming and creating hierarchies within the primary school, where the different groups received separate study materials and had separate examinations. She said that the approaches of teaching to the three student groups were so different that they were almost taught different mathematics. Students from the Pratibha group were following Math Magic curriculum of NCERT, doing estimation, pattern recognition, and learning through multiple rich contexts. Whereas the students from Neo Nishtha were only working on the worksheets, reciting number charts and drilling procedures. Sethi recalls from her experience that it took time for teachers to believe in NCERT’s Math Magic textbooks that presented the alternative and more meaningful ways of doing Mathematics. However, using the mechanical learning worksheets, now she thinks that the teachers are doubting all the work they did as part of using Math Magic? textbooks. Sethi finds that this already chaotic situation for developing foundational numeracy is going to be worse when the existing 5 years time will be reduced to 3 years. The time of 5 years was essential as it enabled teachers to address students' cognitive and other social needs. Sethi pointed out policy's suggestion that state examination at the end of class 3 would be conducted by authorities outside of school, and not by the teachers or teacher representatives from the department like earlier. This she thinks will increase pressure on parents and students, and push teachers towards teaching to test. She also wondered how the state examination will fit into the earlier stated policy of "no detention policy". According to Sethi, making mathematics available at two levels and introducing coding in middle grades, all these efforts suggested in NEP-2020, seem to be creating further divide between who can or who cannot not do math. Lastly, she noted her disappointment that the NEP-2020 does not treat her (a math teacher) as a professional, but rather treats her as an end-user of various teaching apps and private advisory bodies.

## Multidisciplinary Higher Education in NEP-2020

Dr. Tulsi Srinivasan is a faculty member in Mathematics at the Azim Premji University. She finds the suggestions for Higher Education (HE) given in NEP-2020 requiring an overhaul of the system, which would require deliberation on several issues, such as how HE is regulated, role of state vs that of centre, teacher training, and many others. However, Srinivasan said that one specific suggestion in NEP-2020 that has great potential to be useful and relevant is the introduction of multidisciplinary degree for 4 years in mathematics. She cautioned the audience that multidisciplinary education is not independent of other contemporary concerns - such as privatisation of education, fee hike and its implication to whom the HE is being provided, etc. Yet, she thinks it is a positive move from the perspective of meaningful learning in HE. Taking the case of Mathematics, she said our graduates hardly know how versatile the subject is, or how it is applied in other subjects and multidisciplinary degree education can provide precisely this exposure. Srinivasan herself worked in places where students from mathematics moved to philosophy, neuroscience, linguistics, finances, etc. Such multi-disciplinary education, would not only help these students at the individual level, but also from the perspective of society and community, she finds such traversing to be essential. She said, in the current world, it is impossible to escape mathematics and technology. And therefore, it is important that people who study math and computer science also study society, history and develop a sense of society's functionality, limitation of mathematics in such contexts or how mathematics can be abused. Srinivasan said, she would be delighted to see her math students entering data journalism, law, and places in which it is increasingly becoming important to have a solid foundation in math. Said this, she agrees that offering multidisciplinary education just for the sake of choice is not worth. Srinivasan believes that students learn not just in the classrooms but by talking to each other in various contexts, such as sports, readings groups, group assignments, etc., and creating such learning spaces, although not easy is essential. Another concern that Srinivasan voiced was about the rising cost of education. She said that this is already happening at the undergraduate level. Giving it visibility in NEP-2020, makes it a plan endorsed by the government, instead of a phenomenon that is happening on its own - and this requires reflection. Srinivasan thought that the policy was successful in articulating the core of what HE can achieve when under "equity and inclusion" section it highlighted that HE is often able to lift individuals and communities out of disadvantage. Considering the proposed agenda in the document, she believes that HE is going to be exclusive, alienating for students and institutions as an act of charity will bring in individuals for their betterment. Whereas, what she believes ideally should have happened that HE becoming a public education, where faculties coming in from different backgrounds, skills and interests and learn from one another. Lastly, she advised as teachers, they need to compensate the interaction aspect of learning, which is not given much of attention in the policy. And therefore constantly need to think about-how math classes can be made more inclusive, opportunities for group work, collaborations, contexts where students are encouraged to talk to each other, etc.

## UGC Norms and NEP-2020

Prof. Amber Habib is Dean of Undergraduate Studies at Shiv Nadar University. He has played an active role in the well known Mathematics Training and Talent Search (MTTS) programme for college students. He has also been involved in educational initiatives that emphasize mathematics together with its applications. Habib reiterated that the NEP-2020 document is difficult to comprehend mainly due to the possibility of multiple interpretations. He said, depending on the reader and her/his agenda, the meanings could be engineered to fit it. He rejects the popular view that the document is "not as bad as it could have been" by situating the policy suggestions in the context of the government's recent actions. Habib said with an honest government, activities suggested in the policy could have happened on the ground without waiting for the final document, and that also includes the funding that India's education sector deserves. The commitment to how much money will be spent on suggested activities was present in the earlier drafts of NEP 2020 and has been lost in the final version. He also remarked that anticipating an increase in Gross Domestic Product (GDP) percentage doesn't make any sense in the current scenario. Habib gave a historical preview of policies that have been shaping HE since 2001. He said University Grant Commission (UGC) - 2001 prescribed specific objectives for HE in regard to multidisciplinary skills, linking general courses with professional ones, developing bridge courses and increasing horizontal and vertical academic mobility. On mathematical thinking he read a quote from UGC-2001 - "today students need a thorough knowledge of fundamental principles, methods, results and a clear perception of the enormous power of mathematical ideas and tools and know-how to use them in all the three phases, viz., modelling, solving and interpreting." Comparing the two policy documents, he deduced that while NEP-2020 repeats some of the aspirations of the 2001 document, it also brings forward a narrow and vocational view of mathematics. He found that the UGC-2001 is a much richer document in terms of its content because it carries voices from diverse groups of faculties (Banaras Hindu University, Delhi University, North-Eastern Hill University, Harish-Chandra University, Jawaharlal Nehru University, Institute of Mathematical Sciences and Kalyani University).

He elaborated on another set of interim policy guidelines provided by the UGC in 2015, referred to as "Choice-based Credit System (CBCS)". This document curtly dismissed the existing system as one which obstructed student choice and mobility. CBCS laid out core objectives as follows: shift from teacher-centric to student-centric education, choice for the number of credits per semester with students, flexible course options for students from across interdisciplinary and intra-disciplinary courses, broad-based education that is at par with global education (e.g. one can combine physics with economics or microbiology with environmental science), and flexibility of multiple entries and exit points across different institutions to complete the degree. Habib said that most of these suggestions had a procedural binding, except the teacher-centric part that he did not understand as a HE description. He found that CBCS promoted student mobility and without explicitly stating so, but as announced by the then HRD minister, also made a push to teach parts of curriculum in a centralised online teaching mode. Habib said this is a consequence of desiring high enrolment rates in higher education but with low investment in infrastructure. However, it needs a uniform curriculum, which has been a strong push from the government since then. Many of the suggestions given in the UGC-2015 are retained in NEP-2020, and therefore Habib said that NEP-2020 is nothing but a continuation of what the ruling government has been planing to do with HE since it came to power.

Citing section 10 of the NEP-2020 document where recommendations for HE start, Habib highlighted the agenda for online teaching, which is justified as the one that will create access for all. Skeptical about this, Habib said, online teaching may create a better access for some, but not for all, and definitely not for some disadvantaged groups. Something that he experienced during the COVID-19 lockdown period, that for no fault of the students, due to the condition prevailing in the country, some students missed out on their education. NEP-2020 also pushes for online teacher training through Swayam and Diksha, and re-iterates the multiple entry and exit for students within a 4-year multidisciplinary degree, which he finds aligned with the recommendations of UGC-2015. NEP-2020 advocates a credit bank, stored centrally, that will aid such students' movement across institutions. NEP-2020 says that the existing regulation for HE has been heavy-handed for decades with a concentration of power within a few bodies, creating a conflict of interest and lack of accountability. Four structures will be set up under a single umbrella called the Higher Education Commission of India (HECI) to address the issues mentioned earlier. These structures replace the University Grant Commission (UGC), although it maintains the four verticals as they were part of UGC. The four verticals are as follows.

1. National Higher Education Regulatory Council (NHERC): Single point regulator for the entire higher education including for teacher education (excluding medical and legal education.)
2. National Accreditation Council (NAC) - A meta-accrediting body that will supervise accreditation by an independent accrediting institution ecosystem.
3. Higher Education Grants Council (HEGC) - Responsible for funding and financing for HE.
4. General Education Council (GEC) - Responsible for setting up learning outcomes, National Higher Education Qualification Framework (NHEQF), and National Skill Qualification Framework (NSQF).

Habib pointed out that flexible "credit transfer" has been an important government goal since the last 5-6 years for Higher Education, even though there is no such demand in the Indian context. The Indian Institutions do not admit any student in the second or third year, so the credit transfer does not make sense in India, as it might for the International Institutions.

Habib clarified that learning outcomes replace larger objectives with testable ones. Unlike UGC, which used to form a uniform curriculum for the institutions, GEC now will enforce expected learning outcomes. Habib thinks GEC's detailed learning outcomes will impact the curriculum development and the testing, as learning outcomes must be testable. The description of learning outcomes will dictate the examination contents, which he said is a severe problem, as this will take us back to "teaching for tests" issue. Habib pointed out the inherent contradictions in NEP-2020. In several sections, the policy says that tests needs to be replaced with more contextual learning, formative assessment, etc. And while such suggestions are vaguely present, the implementation of learning outcome-based frameworks and curriculum is firmly described, which impact content and result in a greater frequency of testing in Higher Education. Similar to these lines, he shared other details he noticed with in the NEP-2020 document. He said the term "multidisciplinary" appears 69 times in the 62 pages, appearing almost on every page, often attached with some other words, a few examples are - education, teaching, learning, colleges, abilities, research, world, teacher education programs, institutions, universities, B.Ed., Higher Education, Undergraduate Education, fields, etc. Similarly "mathematics" appears 18 times and only once on its own - in the paragraph read earlier during the opening remarks. Habib predicted that NEP-2020 would not be replaced anytime soon, and hence suggested the following in response to NEP-2020.

1. As NEP-2020 has left several important terms undefined, a public dialogue is needed on education and mathematics education, what they mean, in an attempt to influence interpretation and execution of the policy.
2. Discussions on meanings of "multidisciplinary" and how much of it can be emphasised.
3. Reflection on whether online and distance learning - is a necessary evil or it has potential to become a full fledged equivalent/substitute for regular education.
4. Figuring out whether mathematics education needs to shift towards the demands of data science and machine learning.
5. Developing a realistic shared understanding of how much can undergraduate education provide in four years - expertise in multiple disciplines to near research level, broad education towards being a fully contributing citizen, including an integrated training for vocation.
6. A reflection on the implications of multiple entries and exit points and standardisation. The current structuring of undergraduate studies involves foundational courses in the first two years and then specialised courses in the last two years of the degree. Allowing the possibility of exit points after the first or

second year, requires unique designing of these years' curriculum, making them independent in themselves.

## Discussion

The panel illustrated the policy content from different angles and concerns. Equity, operationality, assessment, content and its contrast with the ground reality are some of them. The audience raised concerns regarding the dilution of disciplines due to the multidisciplinary approach. The panel thought such dilution is unlikely, especially at the undergraduate level, where there is scope for developing essential techniques and knowledge over the four years. The panel suggested the problems around learning math are actually of a different nature. Students' difficulties are more socio-mathematical – beliefs about math and learning settings, the baggage that students will bring in terms of their environment, language, and gender. Therefore, gaining disciplinary knowledge is not as big a challenge as making students comfortable acquiring this knowledge, and multidisciplinary might actually help in this context. On the other side, the panel also thought that developing such courses just to create a buzz will lead to mediocracy of the curriculum. Rampal cited Delhi University's experience, where courses on communication and foundation skills were added with the disciplinary subject's core courses under the buzz of multidisciplinary. The students treated these as unimportant, mainly because handling a load of extra course was challenging. She said that multidisciplinary is an outdated idea, and we need to look forward to inter-disciplinary efforts. Rampal cited Finland's example, the country that has successfully achieved quality and equity education, moving away from disciplines and making higher education more interdisciplinary. The panel thinks that the policy's emphasis on the multidisciplinary in higher education as an attempt to use bloom's taxonomy when educational research has already rejected its usefulness seems a move away from the creative challenge of designing a relevant inter-disciplinary curriculum. The panel said policies, in general, have an impact on reality and they do NOT (as many would think) just remain on paper. Rampal reminded everyone that 1986 and other policies had impacted curriculum, government schemes, infrastructure and teacher education. She pointed out that NEP is legitimising what has happened in the country's last five years under the current administration's governance. The first draft of NEP as Rashtriya Shiksha Aayog proposed complete centralised control under the PMO office. The opposition and constant discussion have led to it's current status, and as educators and citizens, we need to continue this struggle for equal and justified access to education.

As the topic and discussion of this panel were very timely, it was attended by more than 200 participants during the conference. The panel discussion is also available to view at the youtube channel of MTA (I) - https://www.youtube.com/channel/UCrPMqprzamsWhOiz0uP6o_w.

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# 4 My Digital Evolution in teaching Mathematics during the Corona pandemic by Jyoti Bhola

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## Introduction

Digital learning has been the talk of the town for quite many years now but it really gained momentum when the whole world faced a completely unprecedented crisis at the beginning of this year due to the Corona pandemic. Things became worrisome and schools, colleges and universities were the first to be closed to control the spread of this pandemic. This was quite a rational step on the part of authorities and by mid-March, 2020 all the academic institutions across 61 countries were shut worldwide. This was the time when there was a rush to learn and use online Learning Management Systems. While a handful of tech-savvy teachers found it more convenient to teach their students remotely, a majority of them had to struggle hard to figure out ways to connect to their students digitally. Adopting and adapting to new platforms to continue the learning process was not a cakewalk, either for the teachers or for the students.

There are certain positives and negatives associated with digital learning in general, and in particular for Mathematics too. While learning remotely, one learns at a safe distance sitting in the comfort zone of one's home, information sharing is rapid, there is supposedly a better visualization of things and of course one cuts the cost of commuting, saves paper and everything compliments in an eco-friendly way as compared to the physical mode. On the other hand, there are various obvious negatives, the primary negative being the need of a basic digital device, and robust internet access to teach online.

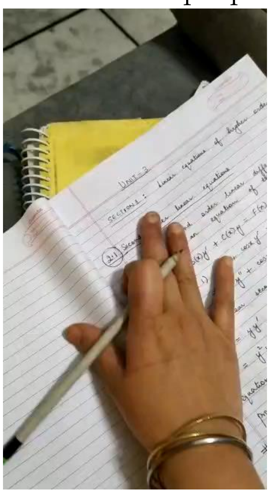
According to an NCERT survey for school students, at least 27% school students do not have access to laptops or smartphones for online classes [1]. The scenario at college or university level is no different. Moreover, adequate electricity supply is a primary requirement to stay connected. Apart from these obvious drawbacks, different surveys and reports suggest that more than these negatives, an extremely significant concern in remote learning is the mental state and unhappiness of students that is dominant. Students are truly missing the class room socialisation and the real connect with their peer group and their mentors [2].

## My Journey so far

While certain restrictions have been eased by the authorities now, schools and colleges are still not back to the normal mode of functioning. Everything is happening online; be it classes, doubts or discussions, assignments or assessment. As a teacher for undergraduate students, various platforms were explored by me during the past couple of months to stay connected to my students. These are listed below with accompanying benefits and challenges in each case.

LEVEL 1:

To be honest, before the Corona pandemic phase, apart from using some Computer Algebra Systems like Mathematica and MATLAB or sometimes power point presentations in my classes, I had never used any digital LMS for my students. The Corona disruption led to a complete repositioning and as a novice I began writing my notes (preferred writing instead of typing due to time constraint), scanning them and sending them to my students. After sharing the pdfs, I used to share small videos of 5-10 minutes duration explaining the concepts shared in the pdfs across on WhatsApp and mail groups dedicatedly created for this purpose of exchange.

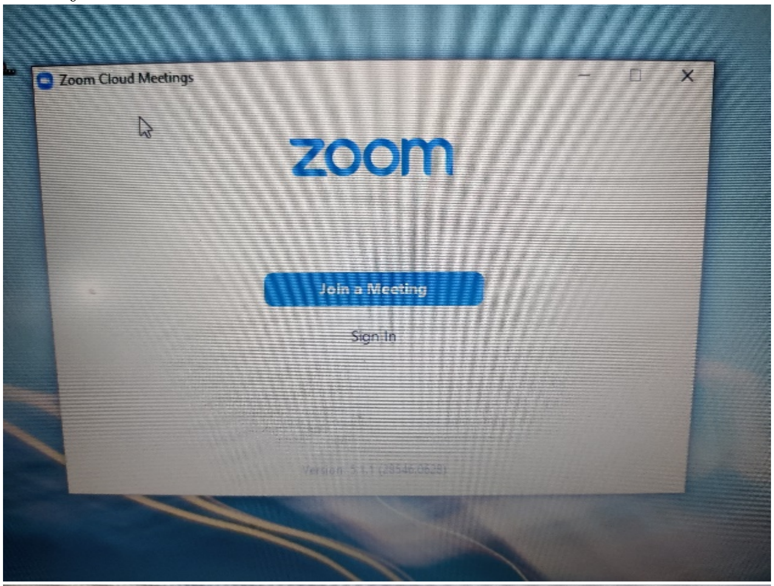


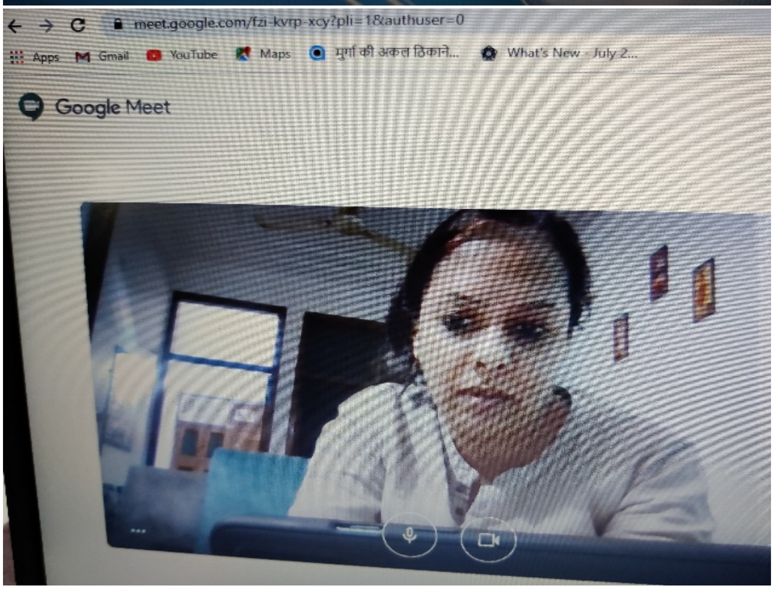
- (Negative) Clumsy videos
- (Negative) Storage issues
- (Negative) Time-consuming transfer
- (Positive) Students' appreciation and understanding

Things worked well with the students this way. In fact, they liked it more as they could access the pdf and the related video at any time as per their convenience. But as a teacher, things became cumbersome for me and the device with which I was capturing the videos simply refused to function, due to the huge size of the videos. Moreover, the transfer of a single such video took a huge quantum of time. So, this raw method had to be discontinued right in its early days and I decided to figure out something better.

LEVEL 2:

Fortunately, this was the time when I got a chance to attend a Faculty Development Programme on Online Teaching and Learning organized by Mahatma Hansraj Faculty Development Centre, MHRD, Govt of India. I learnt about the google meet and google classroom platforms during that programme and started taking my classes using google meet [3] or sometimes Zoom [4] to continue my work.



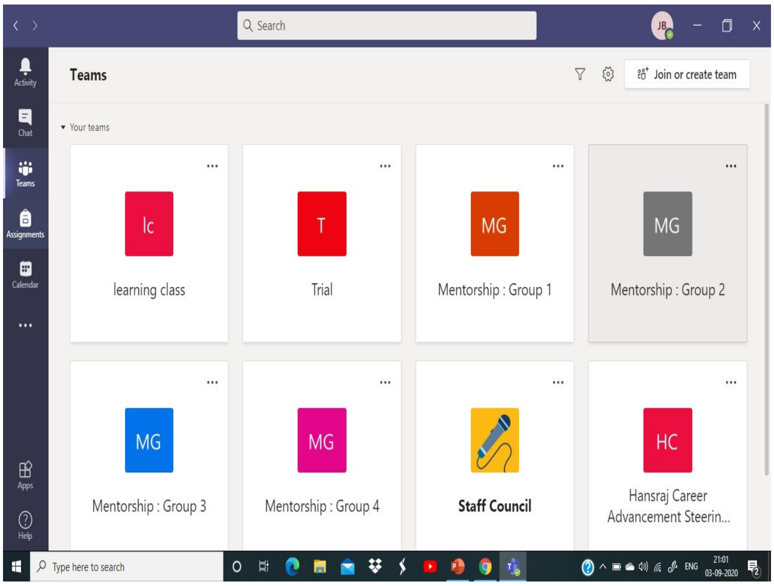




- (Negative) Link of the meeting had to be shared every time
- (Negative) Privacy issues
- (Negative) Getting used to in-built whiteboard takes time
- (Positive) Could record meetings and share the recordings with students for future reference

Things were settled for both the students and I. I got quite comfortable connecting to my students on these platforms, despite a drawback that the students could pass on the meeting link and pass code to outsiders. In a class of around 75 students, it was difficult for me to identify outsiders, if they were present at all. Recording of the class was possible and I used to share the recordings with my students for their future reference. So, things were working quite well, ignoring privacy issues.

LEVEL 3:

Then came the next level of going digital, when our college authorities directed us to use Microsoft Teams [5] for taking classes in the new semester. This wasn't a welcome move for many of us who were quite comfortable with google meet. But we did not have a choice as the authorities wanted a uniformity in this respect. Special sessions were organized by the IQAC Team of our institution to train the faculty members on MS teams and we were all amazed to see the synchronization of features which this platform offers. The platform provides an online class room structure that brings together virtual face to face connections, assignments, files, conversations and so on and at the same time promises to not collect/use data, scan documents, emails and uploads. The administration assigns an official email address to every student and no one else can enter the classroom unless the teacher permits that person to join as a guest. This feature of MS Teams along with a systematic calendar (no need to generate meeting link every time) and download of attendance in Excel sheet format on a single click made it a lot easier for a teacher to carry out day to day teaching related activities.




The use of MS Teams platform synced with OneNote Application along with digital pad and pen made my classes effective and gave me the sense of satisfaction which I was somehow missing earlier.

## Conclusion

Different teachers have experienced different challenges and figured out different solutions to stay connected to their students during these challenging times and when it comes to talking about which platform is the best for teaching and learning Mathematics, there is no obvious winner. The final choice depends on one's requirements, which may vary from topic to topic. Online mode demands new approaches to cultivate student participation and interest. In order to stimulate a meaningful participation, the teachers need to adopt a mix and match strategy to connect to the students in an effective manner. Presently, I am using a blend of all the platforms that I have learnt so far to take my classes, plan assignments, hold meetings and seminars and hope to be more tech-savvy in future. Blended mode is probably the future of education and all of us need to look forward, to accept and move ahead with the desired changes preserving the essence of the traditional mode of education at the same time.

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# 5 Additive problems in number theory by Kaneenika Sinha

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## 5.1 Introduction

The history of number theory is replete with examples of problems which can be explained in simple language to an eager primary school student. However, finding solutions to many such problems have taken centuries of concerted efforts by a wider community of mathematicians (amateur as well as professional). Often, a study of these problems has resulted in deep insights and path-breaking ideas which have revolutionized mathematics. Some of these problems, including those covered in this article, originate from a common source going back several centuries.

Diophantus of Alexandria was a scholar in ancient Greece. Although little is known about his life, extant references indicate that he was born in the early part of the third century CE. He wrote a series of thirteen books titled Arithmetica. The book series is considered the earliest text in Western European history in which mathematical ideas and questions were explained using symbols. They contain several algebraic equations to which Diophantus sought solutions in integers. These textbooks became obscure in Western Europe during the period of the Dark Ages, but some of the books were preserved by Byzantine scholars and were rediscovered in Rome in the fifteenth century. The Arithmetica became available to European scholars when Claude Bachet published a Latin translation of the six surviving books.

Bachet's translation soon gained the attention of mathematics lovers, among whom was a young French lawyer by the name of Pierre de Fermat. Fermat was a lawyer who pursued mathematics as a hobby in his free time, and made significant contributions to the subject. Some of his pertinent observations and insights were written on the margins of his copy of Bachet's translation and were rediscovered by his son a few years after Fermat's passing away. An English translation of one such observation [8, Page 3] is as follows:

"Every number is a triangular number or the sum of two or three triangular numbers; every number is a square or the sum of two, three or four squares; every number is a pentagonal number or the sum of two, three, four or five pentagonal numbers; and so on .... The precise statement of this very beautiful and general theorem depends on the number of angles. The theorem is based on the most diverse and abstruse mysteries of numbers, but I am not able to include the proof here."

The general theorem that Fermat is referring to is about what are called polygonal numbers. The note implies that Fermat had a proof, and it does seem plausible that he did. This theorem was proved in its entirety by Cauchy in 1813. Our focus, however, will be on a specific part of Fermat's note above, namely the conjecture that every natural number can be written as a sum of at most four squares. It is possible that Diophantus was familiar with this conjecture. But, we find the first recorded statement by Bachet in 1621 (in his translation of Arithmetica), which is how Fermat became familiar with it. Bachet also verified it for every number less than 326. In 1748, Leonhard Euler wrote a letter to Christian Goldbach, which contains a fundamental step in the proof of the four square conjecture. This refers to an explicit identity which shows that a product of two numbers, each of which is a sum of at most four squares, is also a sum of at most four squares. Thus, it is sufficient to prove that every prime number can be written as a sum of at most four squares. This was done by Joseph Louis Lagrange in 1770 and the four square theorem is now named after him.

Theorem 1. [Lagrange's four square theorem]
Every natural number is a sum of the squares of at most four natural numbers.

Around the same time, the mathematician Edward Waring, in his book Meditationes Algebraicae conjectured a generalization of the four square theorem. He states that every nonnegative integer is the sum of four squares, nine cubes, fourteenth powers and so on. The phrase "and so on" has the following precise expression.

Conjecture 2. [Waring's Problem, 1770] For each positive integer $k \geq 2$, there exists a positive integer $g = g(k) \geq 2$ such that for any positive integer $n$, there exist $g$ nonnegative integers $x_1, x_2, \ldots x_g$ such that

$$n = x_1^k + x_2^k + \cdots + x_g^k.$$

We note here that $g(k)$ is chosen to be the minimal positive integer with the above property. That is, there exists a natural number $n$ which cannot be written as a sum of $g(k) - 1$ $k$-th powers. Lagrange's theorem is the assertion that $g(2) = 4$. Moreover, as per Waring's conjecture, $g(3) = 9$ and $g(4) = 19$.

As this article proceeds, we will be able to state and interpret Waring's problem in various elegant ways.

Waring's problem leads us to ask two questions: firstly, does $g(k)$ exist for every $k$? Secondly, can we find a precise formula for $g(k)$ for all $k$?

In a parallel correspondence, Euler and Goldbach also discussed fundamental questions about expressing all natural numbers $> 1$ as sums of finitely many primes. In this direction, Goldbach wrote a letter to Leonhard Euler in 1742, in which he made the following conjectures.

Conjecture 3. [Goldbach's binary (strong) conjecture] Every even number $n \geq 4$ can be written as a sum of two primes.

Conjecture 4. [Goldbach's ternary (weak) conjecture] Every odd number $n > 5$ can be written as a sum of three primes.

One sees immediately that the strong conjecture implies the weak one. This follows from the observation that an odd $n > 5$ can be written as $3 + k$, where $k$ is even and $> 2$.

The Goldbach conjectures present an interesting contrast between the additive and multiplicative properties of primes. Recall that by the Fundamental Theorem of Arithmetic, any natural number $n > 1$ can be written uniquely as a product of powers of primes,

$$n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k},\ a_i \geq 1.$$

We now consider a number $c$. Can we write any $n > 1$ as a product of at most $c$ primes? The answer to this question is no. To see this, we note that there are infinitely many primes. Let us denote the $n$-th prime number by $p_n$. Now, for any $c$, we have a number

$$n = p_1 p_2 \dots p_{c+1},$$

which, by the Fundamental Theorem of Arithmetic, cannot be written as a product of at most $c$ primes.

We now give an additive twist to the above question. Can we write $n > 1$ as a sum of at most $c$ primes? Goldbach's conjectures predict that not only do we have an affirmative answer to this question, but also that the value of $c$ is as small as 3.

In one of the most interesting developments in the last decade, the Goldbach ternary conjecture has been proved by Harald Helfgott, a mathematician at the University of Göttingen. However, the binary conjecture is still open.

In this article, we describe various additive problems in number theory related to the conjectures of Goldbach and Waring's problem. We also discuss some notable milestones in the progression of ideas from the earliest attempts to prove the Goldbach conjectures all the way to Harald Helfgott's work. We learn how such problems can be stated in a common "additive" language and an important technique to address them, namely the circle method. Henceforth, our focus will be on Goldbach's conjectures. However, we also make some remarks on Waring's problem.

## 5.2 Journey to the ternary Goldbach conjecture

The study of Goldbach's conjectures can be approached in two different ways. One approach is computational: we check if the conjecture holds for each $n > 1$ one by one. As one can imagine, this becomes more and more intractable as the values of $n$ increase (we will make more remarks about this below).

The other approach is theoretical: it involves using deep mathematical tools which have been carefully developed over many centuries to understand prime numbers. One such tool is the well known Riemann zeta function. The zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for a complex number $s \in \mathbb{C}$ with real part $\mathrm{Re}(s) > 1$. Much of the early work on the zeta function viewed this function for real numbers $s > 1$. However, in a breakthrough treatise written in 1859 [7], Bernhard Riemann studied the zeta function $\zeta(s)$ as a function of a complex variable $s$ and outlined a detailed programme linking the complex-analytic properties of $\zeta(s)$ with the distribution properties of prime numbers. He made two important observations. Firstly, the zeta function can be "analytically extended" to the entire complex plane except the point $s = 1$. Secondly, the zero-free regions of this function (that is, the regions where $\zeta(s) \neq 0$) have a direct bearing on estimates for the prime-counting function $\pi(x)$, defined as the number of primes up to $x$ for large values of $x$.

In this context, Riemann made a conjecture, the well known Riemann hypothesis which predicts that all the non-real zeros of $\zeta(s)$ have real part equal to $1/2$. This conjecture still remains unproved and has motivated a good deal of mathematics over the last 160 years. One of the underlying themes in the study of the zeta function is that as we establish narrower zero-free regions of $\zeta(s)$, we are able to get sharper estimates for $\pi(x)$.

The theme outlined by Riemann can be generalized to series of the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

for various interesting arithmetic functions $f(n)$. An appropriate generalization of the Riemann hypothesis for such functions is called the Generalized Riemann Hypothesis (GRH).

Another important theoretical tool in the study of the Goldbach conjectures is the famous circle method whose origins go back to a seminal 1918 paper of Hardy and Ramanujan [2], in which the earliest form of this method was introduced to study the asymptotic growth for the partition function $p(n)$, that is, the number of ways of writing $n$ as sums of non-negative integers less than or equal to $n$. We note here that the roots of their work further go back to one of the earlier letters that Ramanujan wrote to Hardy from India, which indicates that Ramanujan had a rudimentary form of the circle method in mind. After the unfortunate early demise of Ramanujan, this method was developed by Hardy and Littlewood to address Waring's problem and well as the conjectures of Goldbach. In fact, in 1923, Hardy and Littlewood proved the ternary Goldbach conjecture for "sufficiently large" odd values of $n \geq C$ (under the assumption of the Generalized Riemann Hypothesis). That is, under the condition that the GRH holds, any sufficiently large odd number $n$ can be written as a sum of three primes. This was the first major development in the study of Goldbach conjectures since 1742.

In 1937, Russian mathematican Ivan Vinogradov introduced some remarkably new and beautiful ideas which circumvented the assumption of the Riemann Hypothesis to prove the result of Hardy and Littlewood. He proved the ternary Goldbach conjecture unconditionally for "sufficiently large" odd values $n \geq C$.

Before we proceed further, we make some remarks about the use of the phrase "sufficiently large". The theorems of Hardy-Littlewood and Vinogradov were not able to specify an explicit value of $C$ such that the ternary Goldbach conjecture would hold for all odd $n \geq C$. What they showed was that such a $C$ exists. If one could provide an explicit value of $C$, then one can verify the conjecture for odd $n < C$ through computations and derive a complete proof of the ternary Goldbach conjecture (or disprove it if counterexamples exist). Therefore, three main challenges needed to be overcome before one could complete the treatment of Goldbach's conjectures.

1. Make Vinogradov's theorem effective by establishing an explicit number $C$ such that the ternary Goldbach conjecture holds for $n \geq C$.

2. Verify the conjecture for $n < C$ case-by-case.

3. If $C$ is too large for our current computational resources, then refine $C$ down to a value for which computational verification of $n < C$ is feasible.

These challenges were overcome through multiple developments which are encapsulated below.

- In 1956, K. G. Borodzkin showed that the ternary Goldbach conjecture holds for all
$$n \geq C = 10^{4008659}.$$
- In 1989, J. R. Chen and T. Z. Wang reduced $C$ to $10^{43000}$ and in 1996, to $10^{7194}$.
- In 1997, J. M. Deshouillers, G. Effinger, H. J. te Riele and D. Zinoviev proved the ternary Goldbach conjecture for all odd numbers $n > 5$, but conditionally on GRH.
- $C = 2 \cdot 10^{1346}$ was obtained by M. C. Liu and T. Z. Wang. Until 2013, this remained the lowest known unconditional value for $C$.
- On the most powerful computers, computer verification of the conjecture can be done up to the order of $10^{30}$. In fact, in 2013, Helfgott and D. Platt [6] verified the conjecture for odd $n \leq 8.875 \cdot 10^{30}$.
- In 2013, Harald Helfgott ([3], [4]) proved that the ternary Goldbach conjecture holds for odd $n \geq 10^{27}$. Since the conjecture had already been verified for $n \leq 10^{27}$, Helfgott's result was the proverbial last nail in the coffin that led to a complete proof of the ternary Goldbach conjecture.

### 5.2.1 Related developments

While the above progression of results took place, it was natural to ask the following related question:

Question 5. Take a finite number $D$. Can we write any $n > 1$ as a sum of at most $D$ primes?

This question is a weaker variant of Goldbach's conjectures. If not $D = 3$, can we obtain a larger value of $D$ such that any $n > 1$ is a sum of at most $D$ primes? We now mention some results in this direction.

- In 1930, Lev Schnirelmann showed that such a $D$ exists and is "effectively computable". That is, there is a finite number $D$ such that any $n$ is a sum of at most $D$ primes.
- In 1995, Olivier Ramaré proved that any even $n$ is a sum of at most 6 primes.

- In 2012, Terry Tao proved that any odd $n$ is a sum of at most 5 primes.

We note again that all the above results use the circle method. Before we discuss this method, it would be worthwhile to learn to state additive problems in a new and concise language. We do so in the next section.

## 5.3 Additive problems in the language of Schnirelmann

A fundamental contribution to additive number theory was made by the Russian mathematician Lev Schnirelmann in the 1930s. Schnirelmann was motivated by the conjectures of Goldbach and was able to show that any natural number $n$ can be written as a sum of at most 800000 primes. In order to obtain his theorem, he related the existence of a constant $D$ in Question 5 to a new way of interpreting the density of the set of primes. This new notion of density, different from the notion of "natural" density, can be generalized to subsets of natural numbers and is amenable to several additive problems including the ones posed by Goldbach and Waring. In this section, we introduce the notions of sumsets and Schnirelmann density and how they lead to neat expressions for additive problems in number theory.

We start by defining the notion of sumsets: given two sets $A$ and $B$ of integers, let $A+B$ denote the set $A+B = \{a+b : a \in A, b \in B\}$. Let $\mathbb{N}$ denote the set of natural numbers. If $A$ and $B$ are subsets of the set $\mathbb{N}$, let the sumset $A \oplus B$ be defined as

$$A \oplus B := A \cup \{0\} + B \cup \{0\}.$$

That is,

$$A \oplus B = \{0, a, b, a+b : a \in A, b \in B\}.$$

The above definition can be extended inductively to the sumset $\sum_{i=1}^{m} A_i$ of a finite number of sets. In particular, we consider $A_i = A$ for each $i$ and define, for each positive integer $m$

$$mA := (A \oplus A \oplus \cdots \oplus A) \cap \mathbb{N}.$$

In this notation, a typical additive problem asks if $mA = \mathbb{N}$ for some $m \in \mathbb{N}$. Here, $A$ is a specified subset of natural numbers.

In the context of Goldbach's conjectures, the set of interest to us is the set of all primes, which we denote as $\mathbb{P}$. The ternary Goldbach conjecture states that $3\mathbb{P} = \mathbb{O}_{>1}$ where as the binary Goldbach conjecture states that $2\mathbb{P} = \mathbb{E}$, where $\mathbb{O}$ and $\mathbb{E}$ denote the sets of odd and even positive integers respectively.

In the context of Waring's problem, we are interested in the sets

$$B_k := \{n^k : n \in \mathbb{N}\},$$

where $k \geq 2$ is a fixed positive integer. Waring's problem asks if, for each $k$, one can find a natural number $g = g(k)$ such that $gB_k = \mathbb{N}$.

As mentioned above, Schnirelmann approached these problems with the help of a new notion of density, which we now define.

Let $A \subseteq \mathbb{N}$ and for every $n \geq 1$, let $A(n) = \#\{a \in A : a \leq n\}$. The Schnirelmann density of $A$, denoted by $\delta(A)$ is defined as

$$\delta(A) := \inf{}_{n\geq 1} \frac{A(n)}{n}.$$

We observe that $A(n) \geq \delta(A)n$ for all $n \geq 1$. We also observe that $0 \leq \delta(A) \leq 1$ and $\delta(A) = 1$ if and only if $A = \mathbb{N}$.

The Schnirelmann density is different from the natural density or asymptotic density $\sigma(A)$ defined as

$$\sigma(A) = \lim_{n\to\infty} \frac{A(n)}{n}.$$

While $\sigma(A)$ measures the asymptotic behaviour of $\frac{A(n)}{n}$ for arbitrarily large values of $n$, the Schnirelmann density $\delta(A)$ is also sensitive to small values of $n$. For example, $\sigma(\mathbb{E}) = \sigma(\mathbb{O}) = 1/2$. On the other hand, $\delta(\mathbb{E}) = 0$ and $\delta(\mathbb{O}) = 1/2$.

One of the most fundamental contributions of Schnirelmann was to observe that if $\delta(A) > 0$, then $mA = \mathbb{N}$ for some positive integer $m$. We note here that this observation is not directly applicable to $\mathbb{P}$ and $B_k$ since these sets have Schnirelmann density 0. One shows, instead, that $\delta(\{1\} \cup 2\mathbb{P}) > 0$ and that there exists $l \in \mathbb{N}$ such that $\delta(lB_k) > 0$. We can now apply Schnirelmann's observation to these modifed sets.

The observation that $\delta(\{1\} \cup 2\mathbb{P}) > 0$ leads to the conclusion that

$$D(\{1\} \cup 2\mathbb{P}) = \mathbb{N} \text{ for some } D \in \mathbb{N},$$

and therefore, the answer to Question 5 is affirmative. In particular, in the context of Goldbach's conjectures, Schnirelmann's work can be made effective: one can show that

$$800000\mathbb{P} = \mathbb{N}_{>1},$$

that is, any natural number $> 1$ can be expressed as a sum of at most 800000 primes. The journey to reduce the number of primes from 800000 to 3 requires much finer analysis, but Schnirelmann's theorem was the first explicit answer to Question 5, as it were. The results of Section 5.2.1 can now be restated in Schnirelmann's language as follows:

- Let $\mathbb{P}$ be the set of primes. Schnirelmann showed that there exists a natural number $D$ such that

$$D\mathbb{P} = \mathbb{N}_{>1}.$$

His calculations, in fact, show that $D < 800{,}000$.

- Ramaré (1995): $6\mathbb{P} = E$.
- Tao (2012): $5\mathbb{P} = O_{>1}$.

Coming back to Waring's problem, in 1943, Yuri Linnik showed that there exists $l \in \mathbb{N}$ such that $\delta(lB_k) > 0$. This leads to the conclusion that

$$m(lB_k) = \mathbb{N} \text{ for some } m \in \mathbb{N}.$$

Linnik's theorem solves Waring's problem (Conjecture 2) as we can now take $g = ml$. However, this theorem does not provide effective estimates for $g(k)$. This aspect was investigated by Hardy and Littlewood using the circle method. This method which will be the focus of the next few sections.

## 5.4 The language of exponential sums

In this section, we describe additive problems in terms of calculating integrals involving the exponential function. This interpretation of such problems lies at the heart of most of the work done on the Goldbach conjectures as well as Waring's problem.

For a real number $x$, let

$$e(x) := e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x.$$

Note that $e(x)$ is a periodic function of period 1. The method of exponential sums originates in the following integral identity: for an integer $m$,

$$\int_0^1 e(mx)dx = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0. \end{cases} \tag{1}$$

To use this identity for our purposes, let us take a natural number $n$ and define the exponential sum

$$f(x) := \sum_{\substack{p \le n \\ p \text{ prime}}} e(px).$$

Then,

$$f^2(x) = \sum_{\substack{p_1, p_2 \le n \\ p_1, p_2 \text{ primes}}} e((p_1 + p_2)x).$$

Thus,

$$\int_0^1 f^2(x)e(-nx)dx = \sum_{p_1, p_2 \le n} \int_0^1 e((p_1 + p_2 - n)(x))dx.$$

Using the integral identity (1), we deduce that

$$\int_0^1 f^2(x)e(-nx)dx = \#\{\text{primes } p_1, p_2 \leq n : \, p_1 + p_2 = n\}.$$

The binary Goldbach conjecture now reduces to the following question.

Question 6. Can we show that $\int_0^1 f^2(x)e(-nx)dx > 0$ for all even $n \geq 4$?

Similarly, to address the ternary Goldbach conjecture, our object of interest is

$$\int_0^1 f^3(x)e(-nx)dx.$$

The ternary Goldbach conjecture reduces to the following question.

Question 7. Can we show that $\int_0^1 f^3(x)e(-nx)dx > 0$ for all odd $n > 5$?

The weaker versions of the Goldbach conjectures can be summarised in the following intermediate question.

Question 8. Does there exist a positive number $D$ such that

$$\int_0^1 f^D(x)e(-nx)dx > 0$$

for (sufficiently large) $n$ ?

From what we learnt in the previous section, Schnirelmann's answer to the above question is affirmative and his calculations give us a large, but explicit value, $D = 800000$. Vinogradov's result provides an affirmative answer to Question 8 for $D = 3$ and "sufficiently large" odd values of $n$. But, it is "ineffective", that is, unable to specify how large is sufficiently large.

Theorem 9 (Vinogradov). For sufficiently large odd values of $n$,

$$\int_0^1 f^3(x)e(-nx)dx > 0.$$

Helfgott's theorem ([3], [4]) is the sharpest known effective version of Vinogradov's theorem.

Theorem 10 (Helfgott). For odd values of $n \geq 10^{27}$,

$$\int_0^1 f^3(x)e(-nx)dx > 0.$$

As we discussed in Section 5.2, by combining Helfgott's theorem with computer verification of Goldbach's conjecture up to $10^{27}$, the ternary Goldbach conjecture is proved. In the coming sections, we explore how the above integrals can be evaluated.

## 5.5 Origins of the circle method

Let us consider a subset $\mathcal{A}$ of natural numbers. Additive questions described in this article can be modified into the following general form.

Question 11. Does there exist $g \in \mathbb{N}$ such that every $n \in \mathbb{N}$ can be written as a sum of at most $g$ elements in $\mathcal{A}$?

More precisely, let $r_{\mathcal{A},g}(n)$ be defined as

$$r_{\mathcal{A},g}(n) := \#\left\{(x_1, x_2, \ldots, x_g) : x_i \in \mathcal{A} \cup \{0\}, \sum_{i=1}^{g} x_i = n\right\}.$$

Does there exist a natural number $g$ such that $r_{\mathcal{A},g}(n) > 0$ for each $n \in \mathbb{N}$ or, at least, for sufficiently large values of $n$?

A related question to ask is if we can find an exact formula for $r_{\mathcal{A},g}(n)$ for given $g$, $n \in \mathbb{N}$. Alternatively, for each $g$, can we determine the asymptotic growth of $r_{\mathcal{A},g}(n)$ as $n \to \infty$? In 1918, G. H. Hardy and S. Ramanujan [2] studied the above questions through a complex-analytic approach which is referred to as the circle method. They consider the power series

$$P(z) = \sum_{a \in \mathcal{A} \cup \{0\}} z^a, \ z \in \mathbb{C}.$$

We observe that $P(z)$ converges absolutely when $|z| < 1$. Thus, in this region,

$$(P(z))^g = 1 + \sum_{n=1}^{\infty} r_{\mathcal{A},g}(n) z^n,$$

where, by Cauchy's residue theorem,

$$r_{\mathcal{A},g}(n) = \frac{1}{2\pi i} \int_{C_0} \frac{(P(z))^g}{z^{n+1}} dz.$$

Here $C_0$ refers to the closed circular contour $\{z \in \mathbb{C} : |z| = \rho\}$ for some $0 < \rho < 1$.

Ramanujan and Hardy used this representation to study the asymptotic growth of the partition function $p(n)$, which denotes the number of representations of $n$ as a sum of natural numbers less than or equal to $n$. In this case, we simply take $\mathcal{A} = \mathbb{N}$.

The key idea in evaluating the integral

$$\frac{1}{2\pi i} \int_{C_0} \frac{(P(z))^g}{z^{n+1}} dz$$

is to focus around points on $C_0$ where $|P(z)|^g$ is expected to be very large. In this context, a formula of Euler tells us that

$$P(z) = \prod_{n=1}^{\infty} \frac{1}{1 - z^n}, \ |z| < 1.$$

This formula tells us that the above integral takes large values as $z$ comes close to a root of unity, that is, as $z$ comes close to complex numbers of the form $e(a/q)$, where $a/q$ are rational numbers.

Subsequently, in order to overcome issues concerning convergence of the generating series $P(z)$, Vinogradov observed that one may replace $P(z)$ by a finite sum

$$p(z) := \sum_{\substack{a \in \mathcal{A} \cup \{0\} \\ a \leq n}} z^a.$$

In this case,

$$p(z)^g = \sum_{m=0}^{gn} r_{\mathcal{A},g}^{(n)}(m) z^m,$$

where

$$r_{\mathcal{A},g}^{(n)}(m) := \# \left\{ (x_1, x_2, \ldots, x_g) : x_i \in \mathcal{A} \cup \{0\},\, x_i \leq n,\, \sum_{i=1}^{g} x_i = m \right\}.$$

Since $p(z)$ is a polynomial and therefore analytic on the entire complex plane, we immediately deduce that

$$r_{\mathcal{A},g}^{(n)}(m) = \frac{1}{2\pi i} \int_C \frac{(p(z))^g}{z^{m+1}} dz,$$

where $C$ refers to the closed circular contour $\{z \in \mathbb{C} : |z| = 1\}$. We also observe that if $m \leq n$, then $r_{\mathcal{A},g}^{(n)}(m) = r_{\mathcal{A},g}(m)$. Thus,

$$r_{\mathcal{A},g}(n) = \frac{1}{2\pi i} \int_C \frac{(p(z))^g}{z^{n+1}} dz.$$

We now make the substitution $z = e(x)$ and deduce,

$$r_{\mathcal{A},g}(n) = \int_0^1 (F(x))^g e(-nx) dx,$$

where

$$F(x) = \sum_{\substack{m \in \mathcal{A} \cup \{0\} \\ m \leq n}} e(mx).$$

This exponential sum is similar to the sum $f(x)$ studied in the previous section, with the key difference that we also consider $m = 0$ in the sum defining $F(x)$.

In fact, for the study of Goldbach conjectures, we study the integral

$$\int_0^1 f^D(x) e(-nx) dx, \quad f(x) = \sum_{\substack{m \in \mathbb{P} \\ m \leq n}} e(mx),$$

whereas for Waring's problem, we study

$$r_{B_k,g}(n) = \int_0^1 F^g(x) e(-nx) dx, \quad F(x) = \sum_{\substack{m \in B_k \cup \{0\} \\ m \leq n}} e(mx).$$

As indicated in the discussion above, we are motivated to focus on small neighbourhoods of rational numbers in $[0, 1]$ where the concerned integrals would take unusually large values. We now try to describe these ideas in the context of the Goldbach conjectures.

### 5.5.1 The circle method and the Goldbach Conjectures

The most fundamental observation in the application of the circle method to the Goldbach conjectures is that the function $f(x) = \sum_{p \leq n} e(px)$ peaks at rational numbers $x = a/q$ with suitably bounded denominators. So, we partition the interval $[0, 1]$ into two parts: the major arcs $\mathfrak{M}$, which are unions of very tiny intervals around the rational numbers at which the function peaks and the minor arcs

$$\mathfrak{m} = [0, 1] \setminus \mathfrak{M},$$

which are the portions left behind in the unit interval after taking away the major arcs.

Remark 12. Before we proceed, we introduce some notation that helps us to describe how an arithmetic function behaves for large values of $n$.

Let $F : \mathbb{R} \to \mathbb{C}$ be a complex-valued function and $G : \mathbb{R} \to \mathbb{R}$ be a real-valued function. In addition, we assume that $G(x)$ is strictly positive for sufficiently large values of $x$.

1. We say that $F(x) = \mathrm{O}(G(x))$ or $F(x) \ll G(x)$ if we can find a positive real number $C$ and a real number $x_0$ such that $|F(x)| \leq C\, G(x)$ for all $x \geq x_0$. The constant C is called the implied constant. On a related note, we say that

$$F(x) = H(x) + \mathrm{O}(E(x)) \text{ if } F(x) - H(x) = \mathrm{O}(E(x)).$$

2. We denote $F(x) \approx G(x)$ to mean

$$F(x) = G(x) + \mathrm{O}(1).$$

   In other words, the difference between $F(x)$ and $G(x)$ is practically negligible for large values of $x$.

3. We say that $F(x) \sim G(x)$ as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{F(x)}{G(x)} = 1.$$

We now evaluate the sum

$$f(x) = \sum_{\substack{p \le n \\ p \text{ prime}}} e(px)$$

at rational values of $x$. The most obvious values to start with are $x = 0, 1$ and one immediately sees that $f(0) = f(1) = \pi(n)$. For $x = 1/2$, we observe that

$$f(1/2) = \sum_{p \le n} e^{i\pi p} = \sum_{p \le n} (-1)^p \approx -\pi(n).$$

To evaluate $f(1/3)$, we note that if $p = 3l + r$, then $e(p/3) = e(l/3)$. Thus,

$$f(1/3) = \sum_{p \le n} e(p/3) = \sum_{r=1}^{2} e(r/3) \sum_{\substack{p \le n \\ p \equiv r\ (\mathrm{mod} 3)}} 1 \approx (-1)\frac{\pi(n)}{2}.$$

More generally, for $q \ge 1$, if a prime $p \equiv r \pmod q$, then $(r, q) = 1$ since $p = lr + q$ is a prime. Thus, for a rational number $a/q$ with $a < q$,

$$f(a/q) = \sum_{p \le n} e(pa/q) = \sum_{\substack{r \bmod q \\ (r,q)=1}} e(ra/q) \left( \sum_{\substack{p \le n \\ p \equiv r\ (\mathrm{mod}\ q)}} 1 \right).$$

A very deep theorem of Siegel and Walfisz about the distribution of prime numbers in arithmetic progressions tells us that for $r$ with $(r, q) = 1$, the inner sum above is equal to

$$\sum_{\substack{p \le n \\ p \equiv r\ (\mathrm{mod}\ q)}} 1 = \frac{\pi(n)}{\phi(q)} + \mathrm{O}\left( \frac{\pi(n)}{(\log n)^B} \right) \text{ for any } B > 0.$$

Here, $\phi(q)$ denotes the Euler $\phi$-function, which counts the number of coprime residue class modulo $q$ and the implied constant in the error term depends on $B$. In fact, the Siegel-Walfisz theorem is precisely where we need information about the zero free regions of certain generalized types of zeta functions, thereby bringing in the complex-analytic perspective introduced by Riemann. Thus,

$$f(a/q) = \sum_{\substack{r \bmod q \\ (r,q)=1}} e(ra/q) \left( \frac{\pi(n)}{\phi(q)} + \mathrm{O}\left( \frac{\pi(n)}{(\log n)^B} \right) \right) \text{ for any } B > 0.$$

With some more work, we eventually get

$$f(a/q) = \frac{\pi(n)}{\phi(q)}\mu(q) + \mathrm{O}\left( \frac{\pi(n)}{(\log n)^{B'}} \right) \tag{2}$$

for an appropriate $B' > 0$. Here, $\mu(q)$ denotes the Möbius function, defined as

$$\mu(q) = \begin{cases} 1 & \text{if } q \text{ is squarefree and has an even number of prime factors} \\ -1 & \text{if } q \text{ is squarefree and has an odd number of prime factors} \\ 0 & \text{if } q \text{ is not squarefree.} \end{cases}$$

Equation (2) is extremely important for us to isolate the major arcs in the integrals $\int_0^1 f^k(x)e(-nx)dx$ for $k = 2, 3$. In fact, (2) indicates that $f(a/q)$ will take larger values for rational numbers $a/q$ with "small " values of $q$ as compared with those rational numbers with large denominators.

We now compare the values of $f(a/q)$ with the "average" size of $f(x)$ on $[0, 1]$. Note that the average size of $|f(x)|^2$ is approximately

$$\begin{aligned}\int_0^1 |f(x)|^2 dx &= \int_0^1 f(x)\overline{f(x)}dx \\ &= \int_0^1 \sum_{p_1,p_2 \le n} e(p_1 x)e(-p_2 x)dx \\ &= \sum_{p_1,p_2 \le n} \int_0^1 e(p_1 x - p_2 x)dx = \pi(n).\end{aligned}$$

This indicates to us that the average size of $f(x)$ is approximately $\sqrt{\pi(n)}$, where as, by equation (2), the value of $f(x)$ at a rational point with a not-too-big denominator is close to $\pi(n)$. Therefore, such rational numbers are outliers. The key points of the circle method as applied to the Goldbach conjecture are now briefly summarized.

1. After specifying the bounds for the values of $q$, we partition the integral into integrals over major arcs and minor arcs and write

$$\int_0^1 f^k(x)e(-nx)dx = \int_{\mathfrak{M}} f^k(x)e(-nx)dx + \int_{\mathfrak{m}} f^k(x)e(-nx)dx,\ k = 2,\ 3.$$

2. The major arcs are unions of short intervals around rational numbers $a/q$ with denominators $q$ bounded by a suitable function of $n$. The integral over the major arcs is expected to give the dominant contribution to the integral over the unit interval. We call this the "main term". For the major arcs, as discussed above, $f(x)$ is much larger than the average size of $|f(x)|$ on the unit interval. However, the length of the arcs are small.

3. The integral over the minor arcs gives the "error term", that is, a term of much smaller order than the main term. Here, $f(x)$ is relatively smaller, but the length of the arcs are big. The evaluation of the integral over the minor arcs is a very intense calculation with cutting edge techniques from the theory of exponential sums and Diophantine approximation theory.

4. Unfortunately, in the case of the binary Goldbach conjecture, in order to get an error term of smaller order than the main term, we would require $\max_{x \in \mathfrak{m}} |f(x)|$ to be of smaller order than $\sqrt{\pi(n)}$, the average size of

$|f(x)|$ on $\mathfrak{m}$. That, we need the maximum value of $|f(x)|$ in the minor arcs to be smaller than the average size of $|f(x)|$ in that region, which is clearly not possible. Therefore, the circle method fails for the binary Goldbach conjecture.

5. This problem does not occur if we apply the circle method to the ternary Goldbach problem. This leads to the first partial, but unconditional result in the ternary Goldbach conjecture, which we state below.

Theorem 13 (Vinogradov, 1937). There exists an arithmetic function $\mathcal{G}(n)$ and positive constants $A$ and $B$ such that

$$A < \mathcal{G}(n) < B$$

for all sufficiently large odd integers $n$ and

$$\int_0^1 f(x)^3 e(-nx)dx = \mathcal{G}(n)\frac{n^2}{2} + \mathrm{O}\left(\frac{n^2}{(\log n)^c}\right)$$

for a positive constant $c$.

In the context of Question 7, we ask if Vinogradov's theorem can be made effective enough such that

$$\int_0^1 f(x)^3 e(-nx)dx > 0.$$

We note here that by the above theorem,

$$\int_0^1 f(x)^3 e(-nx)dx \geq \mathcal{G}(n)\frac{n^2}{2} - R\frac{n^2}{(\log n)^c}.$$

To have

$$\mathcal{G}(n)\frac{n^2}{2} - R\frac{n^2}{(\log n)^c} > 0,$$

it is sufficient to show

$$\frac{A}{2} - \frac{R}{(\log n)^c} > 0.$$

That is,

$$n > e^{(2R/A)^{1/c}}.$$

Thus, if one was able to carefully record constants in all the estimates that lead to Vinogradov's theorem, one has an explicit bound $C$ such that the ternary conjecture holds for all $n > C$. But, this is not easy! As indicated in Section 5.2, the journey to a complete proof of the ternary conjecture is a careful progression of ideas that helps us to refine the bounds for $C$ down to a level such that it is computationally feasible to check the conjecture for values up to $C$. Helfgott's theorem makes Vinogradov's theorem effective enough to give us the ternary Goldbach conjecture. He obtains $C = 10^{27}$. A numerical verification for $n \leq 8.875 \cdot 10^{30}$ has been done by Helfgott and D. Platt [6]. Thus, the ternary Goldbach conjecture is proved.

## 5.6 The circle method and Waring's problem

We now make some remarks on the application of the circle method to the other additive problem described in this article, namely Waring's problem.

Let $k \geq 2$ and

$$r_{g,k}(n) := \#\{(x_1, x_2, \ldots, x_g) : x_i \in \mathbb{N} \cup \{0\},\, x_1^k + x_2^k + \cdots + x_g^k = n\}.$$

If $P = n^{1/k}$ and

$$F(x) = \sum_{0 \leq m \leq P} e(m^k x),$$

then, using the integral identity (1), we have

$$r_{g,k}(n) = \int_0^1 F(x)^g e(-nx)dx. \tag{3}$$

In a series of articles written between 1920 and 1928 (see, for example, [8, Chapter 5]), Hardy and Littlewood applied the circle method to the evaluation of the integral above. As in the treatment of Goldbach problem, one has to isolate the major and minor arcs and derive values of $g = g(k)$ for which the main term will dominate the error term, leading to a positive value for $r_{g,k}(n)$ for (sufficiently large) values of $n$.

In fact, Hardy and Littlewood wrote a series of papers culminating in [1], in which they showed that for $g > 2^k$, one can obtain asymptotics for $r_{g,k}(n)$ for large values of $n$.

Theorem 14 (Hardy-Littlewood, 1920-1928). Let $k \geq 2$ and $g \geq 2^k + 1$. There is a $\delta > 0$ such that

$$r_{g,k}(n) = \frac{\Gamma(1 + 1/k)^g}{\Gamma(g/k)} n^{\frac{g}{k}-1} \mathcal{G}(n) + \mathrm{O}\left(n^{g/k-1-\delta}\right).$$

Here,

- $\Gamma(s)$ is the $\Gamma$-function given by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

- $\mathcal{G}(n)$ is an arithmetic function and $A$ and $B$ are positive constants such that $A < \mathcal{G}(n) < B$ for all $n \geq 1$.

As a consequence, we deduce that for $g > 2^k + 1$,

$$r_{g,k}(n) \sim \frac{\Gamma(1 + 1/k)^g}{\Gamma(g/k)} n^{\frac{g}{k}-1} \mathcal{G}(n) \text{ as } n \to \infty.$$

## 5.7 Conclusion and suggestions for further reading

We hope that we were able to give the reader some insights into additive problems in number theory as well as the circle method that originated in the work of Hardy and Ramanujan [2]. If the reader wishes to acquire more knowledge about the topics discussed in this article, we refer them to [8], which contains an elaborate and eminently instructive coverage of important topics in additive number theory. To the eager and curious reader who wishes to work through the proof of Helfgott's celebrated result, we suggest Helfgott's treatise [5], in which a complete and self-contained proof of the ternary Goldbach conjecture is presented.

Finally, we remind the reader that the binary Goldbach conjecture is still open. A careful study of Vinogradov's application of the circle method to the ternary Goldbach conjecture (see [8, Chapter 8]) helps us to understand why this method stops short of conquering the binary Goldbach conjecture.

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# 6 Brouwer and Galerkin by S.Kesavan

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Abstract

We look at an equivalent form of the Brouwer fixed point theorem and explain how it can be used in the Galerkin method for solving equations in Hilbert spaces.

## 6.1 The Brouwer fixed point theorem

The two well-known fixed point theorems are the contraction mapping theorem and the Brouwer fixed point theorem. While the former is fairly easy to prove, the latter needs a sophisticated topological tool, viz. the topological degree.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We denote its closure by $\overline{\Omega}$ and its boundary by $\partial\Omega$. If $f : \overline{\Omega} \to \mathbb{R}^N$ is a given mapping, a vector $b \in \mathbb{R}^N$ is said to be a boundary value if there exists $x \in \partial\Omega$ such that $f(x) = b$.

Let $\Omega$ be as stated above and let $f : \overline{\Omega} \to \mathbb{R}^N$ be a continuous mapping. Let $b \in \mathbb{R}^N$. If $b$ is not a boundary value, then the (Brouwer) degree, denoted $d(f, \Omega, b)$ can be defined (cf. Kesavan [2]. Amongst the properties of the degree, the following are most useful.

- If $d(f, \Omega, b) \neq 0$, then there exists $x \in \Omega$ such that $f(x) = b$. Thus, this helps in proving the existence of a solution to the given equation.
- (Homotopy invariance) Let $H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^N$ be a continuous mapping such that for every $t \in [0, 1]$, the point $b \in \mathbb{R}^N$ is not a boundary value of the mapping $H(\cdot, t)$, then

$$d(H(\cdot, 0), \Omega, b) \ = \ d(H(\cdot, 1), \Omega, b).$$

- If $I$ stands for the identity mapping in $\mathbb{R}^N$, then

$$d(I, \Omega, b) \ = \ \begin{cases} 1, & \text{if } b \in \Omega, \\ 0, & \text{if } b \in \mathbb{R}^N \backslash \overline{\Omega}. \end{cases}$$

Using these properties, the Brouwer fixed point theorem is proved in two steps, as indicated below.
Let $B$ denote the open unit ball, centered at the origin, in $\mathbb{R}^N$ and let $S^{N-1}$ be its boundary, i.e. the unit sphere.
Step 1. First we prove the 'no retraction theorem': there is no continuous map from from $\overline{B}$ onto $S^{N-1}$ which is the identity map when resticted to $S^{N-1}$.
Step 2: (Brouwer fixed point theorem): Let $f$ be a continuous map of $\overline{B}$ into itself. Then $f$ has a fixed point.

To see this, we assume the contrary. Then, for each $x \in \overline{B}$, the points $x$ and $f(x)$ are distinct and so the line segment joining them is well defined. We produce the line segment starting from $f(x)$ and ending at $x$ to meet the boundary $S^{N-1}$ and call this point, say, $P(x)$. Then one shows that the mapping $x \mapsto P(x)$ is a continuous map (this needs checking) which is, obviously, the identity map when restricted to $S^{N-1}$, thereby arriving at a contradiction.

In one-dimension, the intermediate value theorem states that if $f : [-1, 1] \to \mathbb{R}$ is a continuous function such that $f(-1)$ and $f(1)$ have opposite signs, then $f$ has to vanish in the interval $(0, 1)$. If, now, $f : [-1, 1] \to [-1, 1]$ is continuous, we can apply the intermediate value theorem to the function $f(x) - x$ and immediately deduce Brouwer's theorem in this case.

We now extend this idea to higher dimensions and deduce Brouwer's theorem. We denote by $(\cdot, \cdot)$, the usual euclidean inner-product, and by $|\cdot|$, the euclidean norm in $\mathbb{R}^N$.

Proposition 15. Let $g : \mathbb{R}^N \to \mathbb{R}^N$ be continuous. Let $R > 0$. Assume that $(g(x), x) \geq 0$, for all $x \in \mathbb{R}^N$ such that $|x| = R$. Then there exists $x_0 \in \mathbb{R}^N$ with $|x_0| \leq R$ such that $g(x_0) = 0$.

Proof: Assume that $g(x)$ does not vanish for $|x| = R$, for, otherwise, we are done. Define, for $|x| \leq R$ and for $t \in [0, 1]$,

$$H(x, t) \;=\; tg(x) + (1 - t)x.$$

Then, clearly, $H(x, t)$ does not vanish for $|x| = R$ when $t = 0$ and when $t = 1$. Let $t \in (0, 1)$. If $|x| = R$ and if $H(x, t) = 0$, we hve $g(x) = -\frac{1-t}{t}x$ and so

$$0 \;\leq\; (g(x), x) \;= -\frac{1-t}{t}R^2 \;<\; 0,$$

which is a contradiction. Thus $H(\cdot, t)$ does not vanish on the boundary of the ball $B_R$, centred at the origin and of radius $R$, for all $t \in [0, 1]$. Hence, by the properties of the degree listed earlier, we get

$$d(g, B_R, 0) \;=\; d(I, B_R, 0) \;=\; 1,$$

and so, again by a property of the degree, there exists $x_0 \in B_R$ such that $g(x_0) = 0$. ■

Remark 1.1 We have not only proved the existence of a solution to the equation $g(x) = 0$, but we also have an estimate for the norm of the solution. ■

Remark 1.2 It is clear that the proposition is true if we have the condition $(g(x), x) \leq 0$ for all $x \in \mathbb{R}^N$ such that $|x| = R$. ■

Corollary 16. (Brouwer fixed point theorem) Let $B_R$ be the open ball in $\mathbb{R}^N$, centred at the origin and of radius $R > 0$. Let $f : \overline{B_R} \to \overline{B_R}$ be continuous. Then $f$ has a fixed point.

Proof: Set $g(x) = x - f(x)$. Then, for $|x| = R$, by the Cauchy-Schwarz inequality, we have

$$(g(x), x) \;=\; R^2 - (f(x), x) \;\geq\; 0,$$

since $|f(x)| \leq R$. The result now follows from the preceding proposition. ■

In fact, all the three results - the Brouwer fixed point theorem the above proposition and the no retraction theorem - are all equivalent, i.e. each implies the other two.

Proposition 17. Let $R > 0$ and let $B_R$ denote the open ball in $\mathbb{R}^N$ centred at the origin and of radius $R$. Let $S_R$ denote its boundary, the sphere with centre at the origin and of radius $R$. The following statements are equivalent.
(i) Let $f : \overline{B_R} \to \overline{B_R}$ be continuous. Then $f$ has a fixed point.
(ii) Let $g : \mathbb{R}^N \to \mathbb{R}^N$ be such that $(g(x), x) \geq 0$ for all $x \in S_R$. Then there exists $x_0 \in \overline{B_R}$ such that $g(x_0) = 0$.
(iii) There is no continuous mapping from $\overline{B_R}$ onto $S_R$ which is the identity when restricted to $S_R$.

Proof: (i) $\Rightarrow$ (ii): Assume that $g$ does not vanish in $\overline{B_R}$. Then the mapping

$$f(x) \;=\; -R\frac{g(x)}{|g(x)|}$$

is well-defined, continuous and maps $\overline{B_R}$ into itself. In fact, the image is contained in $S_R$. Thus, by (i), there exists a fixed point, say, $x_0$, of $f$. Then it follows that $x_0 \in S_R$. But then,

$$0 \;<\; R^2 \;=\; |x_0|^2 \;=\; (f(x_0), x_0) \;=\; -R\frac{(g(x_0), x_0)}{|g(x_0)|} \;\leq\; 0,$$

a contradiction.

(ii) $\Rightarrow$ (iii): If $g : \overline{B_R} \to S_R$ is a continuous mapping which, when restricted to $S_R$, is the identity, we have, for all $x \in S_R$,

$$(g(x), x) \;=\; |x|^2 \;\geq\; 0.$$

Thus $g$ must vanish somewhere, which is impossible since it takes values only in $S_R$.

(iii) $\Rightarrow$ (i): This is the standard proof of Brouwer's theorem outlined at the beginning of this section. See Kesavan [2] for details. ■

## 6.2 The Galerkin method

The Galerkin method is a useful approximation method to solve linear and nonlinear equations in Hilbert spaces. In this section, we will illustrate the method by using it to prove the famous Lax-Milgram lemma. In the next section, we will illustrate its use in solving a nonlinear equation and we will use the equivalent form of Brouwer's theorem proved in the previous section.

Let $A$ be an $N \times N$ matrix with real entries. It is said to be positive definite if, for every $x \in \mathbb{R}^N,\ x \neq 0$, we have

$$(Ax, x) \;>\; 0.$$

Remark 2.1 If $A$ is a matrix with complex entries and if we still denote the usual inner-product in $\mathbb{C}^N$ by $(\cdot,\cdot)$, then $(Az, z) = 0$ for all $z \in \mathbb{C}^N$ implies that $A = 0$. Similarly if $(Az, z) \geq 0$ for all $z \in \mathbb{C}^N$, it follows that $A = A^*$, i.e. $A$ is self-adjoint. Neither of these results is true in the real case as seen from the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \;\blacksquare$$

Since the unit ball is compact, we then deduce that, if $A$ is positive definite, there exists $\alpha > 0$ such that

$$(Ax, x) \;\geq\; \alpha|x|^2, \text{ for every } x \in \mathbb{R}^N.$$

It also follows that the linear map associated to $A$ is injective and so $A$ is invertible. The Lax-Milgram lemma is an infinite-dimensional version of this result.

Let $H$ be a real Hilbert space whose norm is denoted by $\|\cdot\|$ and the inner-product by $(\cdot,\cdot)$ and let $a : H \times H \to \mathbb{R}$ be a bilinear form. We say that $a$ is continuous if there exists a constant $M > 0$ such that

$$|a(x,y)| \ \leq \ M\|x\| \ \|y\|,$$

for every $x$ and $y$ in $H$. The bilinear form is said to be elliptic if there exists a constant $\alpha > 0$ such that

$$a(x,x) \ \geq \ \alpha\|x\|^2,$$

for every $x \in H$.

Theorem 18. (Lax-Milgram Lemma) Let $H$ be a real Hilbert space which is separable and let $a : H \times H \to \mathbb{R}$ be a continuous and elliptic bilinear form. Given any $f \in H$, there exists a unique vector $u \in H$ such that

$$a(u,v) \ = \ (f,v),$$

for every $v \in H$.

Proof: Since $H$ is separable, we can find an orthonormal basis $\{w_i\}_{i=1}^{\infty}$. Let $W_m$ be the span of $\{w_1, \cdots, w_m\}$, where $m$ is a positive integer.

Step 1. Let $m$ be a positive integer. Using the notation established above, we consider the following problem: find $u_m \in W_m$ such that, for every $v \in W_m$,

$$a(u_m, v) \ = \ (f,v). \tag{4}$$

By linearity in the second variable of $a$, it is enough for us to find $u_m \in W_m$ such that

$$a(u_m, w_i) \ = \ (f, w_i), \ \text{for } 1 \leq i \leq m.$$

Since we can write

$$u_m \ = \ \sum_{j=1}^{m} u_j^m w_j,$$

we get the system of $m$ linear equations in the $m$ unknowns $\{u_1^m, \cdots, u_m^m\}$,

$$\sum_{j=1}^{m} a(w_j, w_i) u_j^m \ = \ (f, w_i), \ \text{for } 1 \leq i \leq m.$$

The matrix $A = (a_{ij})$ where $a_{ij} = a(w_j, w_i)$ is positive definite. For,

$$(Ax, x) \ = \ \sum_{i=1}^{m} \sum_{j=1}^{m} a(w_j, w_i) x_i x_j \ = a(w,w) \ \geq \ \alpha\|w\|^2,$$

where $w = \sum_{i=1}^m x_i w_i$. Hence, there exists a unique solution $u_m$ to (4).

Step 2. (a priori estimate) Using the ellipticity of the bilinear form $a$, we get

$$\alpha \|u_m\|^2 \;\leq\; a(u_m, u_m) \;=\; (f, u_m) \;\leq\; \|f\|\|u_m\|,$$

whence we get that for all positive integers m,

$$\|u_m\| \;\leq\; \frac{\|f\|}{\alpha}.$$

Since $\{u_m\}$ is uniformly bounded in the Hilbert space $H$, we can extract a weakly convergent subsequence, say, $\{u_{n_k}\}$. Let the weak limit be $u$. Given $v \in H$, let

$$v^{(m)} \;=\; \sum_{i=1}^m v_i w_i,$$

where, $v_i = (v, w_i)$. Then $v^{(m)} \to v$ in $H$, i.e. $\|v^{(m)} - v\| \to 0$. Since $v^{(m_k)} \in W_{m_k}$, we get from (4) that

$$a(u_{m_k}, v^{(m_k)}) \;=\; (f, v^{(m_k)}).$$

The right-hand side of this equation obviously converges to $(f, v)$. By the continuity of the bilinear form, for any fixed $z \in H$, the mapping $x \mapsto a(x, z)$ defines a continuous linear functional on $H$. Hence $a(u_{m_k}, z) \to a(u, z)$, by the definition of weak convergence. Now we have

$$a(u_{m_k}, v^{(m_k)}) \;=\; a(u_{m_k}, v) + a(u_{m_k}, v^{(m_k)} - v).$$

The first term on the right-hand side converges to $a(u, v)$. Since

$$|a(u_{m_k}, v^{(m_k)} - v)| \;\leq\; M\|u_{m_k}\|\|v^{(m_k)} - v\| \;\leq\; \frac{M}{\alpha}\|f\|\|v^{(m_k)} - v\|,$$

it follows that the second term converges to zero. Thus we get that, for all $v \in H$,

$$a(u, v) \;=\; (f, v), \tag{5}$$

whch proves the existence of a solution.

Step 3. (Uniqueness) If we have two solutions $u_1$ and $u_2$, then, for all $v \in H$, we get that $a(u_1 - u_2, v) = 0$. Set $v = u_1 - u_2$. Thus,

$$\alpha\|u_1 - u_2\|^2 \;\leq\; a(u_1 - u_2, u_1 - u_2) \;=\; 0,$$

which shows that $u_1 = u_2$. This completes the proof. ■

Remark 2.1 In fact $\{u_{m_k}\}$ converges to $u$ in norm. For,

$$\begin{aligned}\alpha\|u-u_{m_k}\|^2 &\leq a(u-u_{m_k},u-u_{m_k})\\ &= a(u-u_{m_k},u-u^{(m_k)})+a(u-u_{m_k},u^{(m_k)}-u_{m_k}).\end{aligned}$$

The second term on the right-hand side vanishes since $u^{(m_k)}-u_{m_k}\in W_{m_k}$, in view of (4) and (5). Thus,

$$\|u-u_{m_k}\| \;\leq\; \frac{M}{\alpha}\|u-u^{(m_k)}\|,$$

and the conclusion follows since the right-hand side of the above inequality tends to zero. ■

Remark 2.2 Given any subsequence of $\{u_m\}$, there is a further subsequence which is weakly convergent. The weak limit satisfies (5) which has a unique solution. Therefore, it follows that, in fact, the entire sequence $\{u_m\}$ converges to the unique solution $u$ of (5) in norm. ■

For a proof of this theorem, through the context of variational inequalities, see Kesavan [1]. Incidentally, the proof of existence to a variational inequality uses the contraction mapping theorem. Here, we have proved the Lax-Milgram lemma, directly, in the case of a separable Hilbert space. This is not a serious restriction, since all Hilbert spaces which occur in applications are generally separable.

The Lax-Milgram lemma is the corner stone of the exitence theory for weak solutions of elliptic boundary value problems. For several examples of its use in this direction, see Kesavan [1].

Another example of the Galerkin method to solve a linear problem, without the use of the Lax-Milgram lemma, is the solution of the Schrödinger equation. See Kesavan [1] for details.

## 6.3 A nonlinear equation

We will now illustrate the application of the Galerkin method to solve a nonlinear equation (cf. Kesavan [2]). Let $H$ be a separable Hilbert space whose norm and inner-product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. A mapping $A:H\to H$ is said to be monotone if, for every $x,y\in H$, we have

$$(Ax-Ay,x-y) \;\geq\; 0.$$

We now establish some notation. Let $\{w_i\}_{i=1}^\infty$ be an orthonormal basis of $H$. We denote by $W_m$, the span of $\{w_1, \cdots, w_m\}$, which is an $m$-dimensional subspace of $H$. Let $v \in W_m$. Then

$$v = \sum_{i=1}^{m} (v, w_i) w_i.$$

Set $v_i = (v, w_i)$. Consider the vector $\mathrm{v} \in \mathbb{R}^m$ given by $\mathrm{v} = (v_1, \cdots, v_m)$. Thus we have a linear bijection between $W_m$ and $\mathbb{R}^m$. In fact, since we are working with an orthonormal set, it is immediate to see that this bijection is an isometry, i.e.

$$\|v\|^2 = |\mathrm{v}|^2.$$

Theorem 19. Let $H$be separable Hilbert space. Let $A : H \to H$ be a continuous map which is monotone and which maps bounded sets onto bounded sets. Then, given any $f \in H$, there exists a unique solution $u \in H$ of the equation

$$u + Au = f. \tag{6}$$

Further,

$$\|u\| \leq \|A(0) - f\|. \tag{7}$$

Proof: Step 1. (Uniqueness) If $u_1$ and $u_2$ were two solutions of (6), we have

$$u_1 - u_2 + Au_1 - Au_2 = 0.$$

Thus,

$$\|u_1 - u_2\|^2 + (Au_1 - Au_2, u_1 - u_2) = 0,$$

and hence, by virtue of the monotonicity of $A$, we have $u_1 - u_2 = 0$.

Step 2. (a priori estimate) If $u \in H$ is a solution of (6), then

$$\|u\|^2 + (Au - A0, u) = (f - A0, u),$$

and the estimate (7) follows, again thanks to the monotonicity of $A$.

Step 3. Let $\{w_i\}_{i=1}^\infty$ be an orthonormal basis of $H$. Let $W_m$ be as defined above and for $v \in W_m$ we associate the vector $\mathrm{v} \in \mathbb{R}^m$, as explained above. Define $T : \mathbb{R}^m \to \mathbb{R}^m$ by

$$(T\mathrm{v})_i = (v, w_i) + (Av, w_i) - (f, w_i), 1 \leq i \leq m.$$

Then, $T$ is continuous and (denoting the usual inner product in $\mathbb{R}^m$ by $(\cdot, \cdot)_m$) we have

$$\begin{aligned}(T\mathrm{v}, \mathrm{v})_m &= \|v\|^2 + (Av, v) - (f, v)\\ &= \|v\|^2 + (Av - A0, v) - (f - A0, v)\\ &\geq \|v\|^2 - \|A0 - f\|\|v\|.\end{aligned}$$

Setting $R = \|A0 - f\|$, we have that $(T\mathrm{v}, \mathrm{v})_m \geq 0$ for all $|\mathrm{v}| = \|v\| = R$. Hence, by Proposition 1.1, there exists a $\mathrm{u}_m \in \mathbb{R}^m$ such that $|\mathrm{u}_m| \leq R$ and $T\mathrm{u}_m = 0$. Thus, $u_m \in W_m$ satisfies

$$(u_m, w_i) + (Au_m, w_i) \;=\; (f, w_i), \; 1 \leq i \leq m,$$

i.e.

$$(u_m, v) + (Au_m, v) \;=\; (f, v) \;\text{ for all }\; v \in W_m$$

and $\|u_m\| \leq \|A0 - f\|$.

Step 4. Then, there exists a weakly convergent subsequence of $\{u_m\}$. As $\{Au_n\}$ is bounded (since $A$ maps bounded sets into bounded sets), we can also assume (after taking a further subsequence if necessary) that $u_{n_k} \rightharpoonup u$, $Au_{n_k} \rightharpoonup \chi$ weakly in $H$, for a subsequence indexed by $n_k$.

Step 5. Given $v \in H$, the sequence $\{v^{(n)}\}$ defined, as before, by

$$v^{(n)} \;=\; \sum_{i=1}^{n} (v, w_i) w_i,$$

is such that $v^{(n)} \in W_n$ for each $n$ and $\|v^{(n)} - v\| \to 0$. Now,

$$(u_{n_k}, v^{(n_k)}) + (Au_{n_k}, v^{(n_k)}) \;=\; (f, v^{(n_k)}). \tag{8}$$

Passing to the limit in (8), we get

$$(u, v) + (\chi, v) \;=\; (f, v) \;\text{ for all }\; v \in H. \tag{9}$$

Step 6. By the monotonicity of $A$, we have for any $v \in H$,

$$\begin{aligned} 0 \leq X_{n_k} &= (Au_{n_k} - Av, u_{n_k} - v) \\ &= (Au_{n_k}, u_{n_k}) - (Au_{n_k}, v) - (Av, u_{n_k} - v) \\ &= (f, u_{n_k}) - \|u_{n_k}\|^2 - (Au_{n_k}, v) - (Av, u_{n_k} - v), \end{aligned}$$

using (8). Thus, $X = \limsup_{k \to \infty} X_{n_k} \geq 0$ and

$$\begin{aligned} X &= (f, u) - \liminf_{k \to \infty} \|u_{n_k}\|^2 - (\chi, v) - (Av, u - v) \\ &\leq (f, u) - \|u\|^2 - (f, v) + (u, v) - (Av, u - v), \end{aligned}$$

using (9). Thus,

$$(f - u - Au, u - v) + (Au - Av, u - v) \;\geq\; 0. \tag{10}$$

Let $\lambda > 0$ and $w \in H$. Set $v = u - \lambda w$ in (10) to get

$$(f - u - Au, w) + (Au - A(u - \lambda w), w) \;\geq\; 0.$$

As $\lambda \to 0$, by the continuity of $A$, the second term on the left-hand side tends to zero. Thus $(f - u - Au, w) \geq 0$ for all $w \in H$ and, by considering $-w$ in place of $w$, we conclude that $u$ satisfies (6). ■

For an example of the Galerkin method applied to a semilinear elliptic boundary value problem, see Kesavan [1]. The method is always the same. We study the problem in a finite dimensional subspace. The existence of a solution in that space and the uniform bound for that solution follow from Brouwer's theorem (Proposition 1.1). The sequence of approximate solutions will have a weakly convergent subsequence and the weak limit will turn out to be a solution of the original problem. See also Lions [3] for more examples of this technique.

Thus, while the only serious application of the no retraction theorem seems to be the proof of Brouwer's theorem, Proposition 1.1 has many applications, via the Galerkin method, in the theory of existence of solutions to linear and nonlinear boundary value problems.

## References

[1] Kesavan, S. Topics in Functional Analysis and Applicatios, Third Edition, New Age International Publishers, 2019.

[2] Kesavan, S. Nonlinear Functional Analysis: A First Course, TRIM Series 28, Hindustan Book Agency, 2004.

[3] Lions, J. L. Quelques Méthodes de Résolution des Problèmes aux Limites Nonlinéaires, Dunod Gauthier-Villars, 1969.

# 7 Technology Assisted Exploration – A Case Study by S Muralidharan

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Can technology be used to assist problem solving? We often tell students how a computer assisted proof of Four-Color Theorem was arrived at. Also we awe them about computations of the zeros of the Riemann zeta function. Is it possible to demonstrate this process on a problem that can be understood without much technical background? Specifically, the problem should have the following characteristics:

1. Should require very little Mathematics background to understand
2. Exploration should not involve complicated algorithms
3. Should not take more than half hour to code

One such opportunity presented itself in the following problem that appeared in Indian National Mathematical Olympiad 2019:

Let $A_1B_1C_1D_1E_1$ be a regular pentagon. For $2 \leq n \leq 11$, let $A_nB_nC_nD_nE_n$ be the pentagon whose vertices are the mid points of the sides of the pentagon $A_{n-1}B_{n-1}C_{n-1}D_{n-1}E_{n-1}$. All the 5 vertices of each of the 11 pentagons are arbitrarily colored red or blue. Prove that four points among these 55 points have the same color and form the vertices of a cyclic quadrilateral.

It is obvious that Pigeonhole Principle needs to be used.

## Solution 1

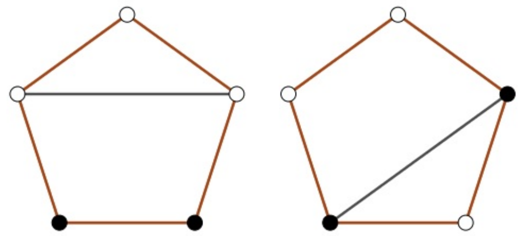
We first observe that all the pentagons are regular. Also, there are five fixed directions and all the 55 sides are in one of these directions. If we consider any two sides which are parallel, they are parallel sides of an isosceles trapezium and hence the end points form a cyclic quadrilateral.

If we consider any pentagon, two of its adjacent vertices have the same color. Consider all such sides whose end points have the same color. There are 11 such lines and these are in 5 directions. Hence by Pigeonhole principle, 3 or more of these are in the same direction and two of these must also have the same colored end points. Hence these two sides form a monochromatic isosceles trapezium.

One of the students gave a proof that six pentagons are enough for the existence of monochromatic isosceles trapezium.

## Solution 2

It is enough to consider the case in which each pentagon is colored with 3 of one color and two of the other. Observe that in any such pentagon, there is a pair of parallel lines with each line having end points of the same color, see the figure below.

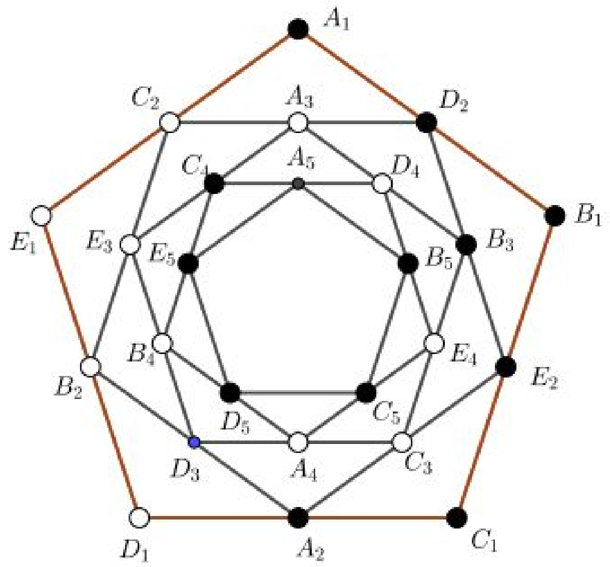


If we have six pentagons, we have six pairs of parallel lines and since we have only five directions, at least two pairs are parallel to the same direction. Further, those lines have the same perpendicular bisector. Thus among the four parallel lines, we have two having the same colored end points forming a monochromatic isosceles trapezium. Now, one can ask: Is six pentagons the minimum number required?

Is there a coloring of vertices of five pentagons such that there is no isosceles trapezium with vertices having the same color? A computer program was written to search for the example with no isosceles monochromatic trapeziums. A brute-force check requires verifying $2^5 \times 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 = 59535360$ cases. Checking this number of cases is a trivial task for computers. The program was written in the R programming language. Any other popular language could have been used. It took only around 10 lines of code. Surprisingly, for any coloring of the vertices, there was always at least one monochromatic isosceles trapezium. The focus shifted to proving that such a monochromatic isosceles trapezium always exists. A clever naming of vertices enabled us to map this problem to a known one about monochromatic rectangles in grids.

## Solution 3

Let us name the vertices of the pentagon as in the figure below.




Note that $X_iY_i \parallel X_jY_j$ for $X, Y \in \{A, B, C, D, E\}$ and $1 \leq i, j \leq 5$. We map the polygons to a $5 \times 5$ grid. The rows are numbered $1, 2, 3, 4, 5$ and columns are numbered $A, B, C, D, E$. In the $i$th row and $X$ column, fill the color of $X_i$. Then we obtain a two colored $5 \times 5$ grid. For example, for the coloring of the pentagons shown in the above figure, the grid formed is given in the figure below.

| | $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|---|
| 1 | $B$ | $B$ | $B$ | $R$ | $R$ |
| 2 | $B$ | $R$ | $R$ | $B$ | $B$ |
| 3 | $R$ | $B$ | $R$ | $B$ | $R$ |
| 4 | $R$ | $B$ | $B$ | $R$ | $R$ |
| 5 | $B$ | $R$ | $B$ | $R$ | $B$ |

If we show that such a grid contains a rectangle all of whose vertices are of the same color, then the corresponding vertices form an isosceles trapezium with all vertices of the same color.

By Pigeonhole Principle, at least 13 squares must have the same color, say, blue. Let $a_1, a_2, \ldots, a_5$ be the number of blue squares in the columns $1, 2, \ldots, 5$

respectively. Then $\sum a_i = 13$. Also,

$$\sum_{i=1}^{5} \binom{a_i}{2} = \frac{1}{2}\left(\sum_{i=1}^{5} a_i^2 - 13\right)$$
$$\geq \frac{1}{2}\left(5\left(\frac{1}{5}\sum a_i\right)^2 - 13\right)$$
$$= \frac{104}{10}$$

Note that we have used the generalized mean inequality,

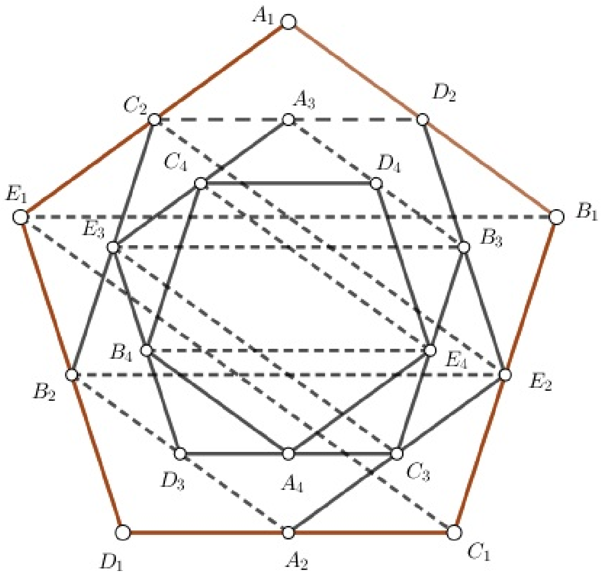
$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \leq \frac{1}{n}\sum_{i=1}^{n} a_i^2$$

Thus

$$\sum_{i=1}^{5} \binom{a_i}{2} \geq 11$$

Since there are only $\binom{5}{2} = 10$ pairs of rows, two of the doublets in the columns determine the same set of two rows. Thus there is a rectangle all of whose corners are blue. Is five the minimum number pentagons required to draw the conclusion? Back to the computer to find an example. But again, the result was negative. Since there is less space to move around among four pentagons (compared to the other cases), we decided to see whether we can crack it by brute force. A simple observation helped us to reduce the number of cases to check from $20 \cdot 19 \cdot 18 \cdot 17 = 116280$ to just 2. We will show that there exists a monochromatic isosceles trapezium even when we have only 4 pentagons.

Let us name the colors $0, 1$. As in Solution 3, we need to fill four rows with $0, 1$ such that in the resulting $4 \times 5$ grid there is no monochromatic rectangle.




Observe that from the above figure some more positions are also forbidden to have the same values. Note the following pairs of parallel lines:

$$A_iB_i \parallel C_jE_j \quad B_iC_i \parallel A_jD_j \quad C_iD_i \parallel B_jE_j \quad D_iE_i \parallel A_jC_j \quad E_iA_i \parallel B_jD_j$$

Since $A_iB_i \parallel C_jE_j$, we can not have identical values in those four positions. Similarly we can not have identical values in the other parallel pairs. Of the twenty possible rows of $0, 1$, consider a row – say – 00011. Using the above figure to eliminate the possible rows that could be the other rows, we find that there are only three possibilities:

10110,  01110,  11001

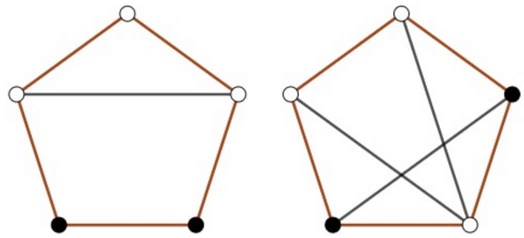
But in this case, the second and third rows contain a monochromatic rectangle. Similarly, if the first row is 01010, then the possibilities are

10110,  01110,  00111

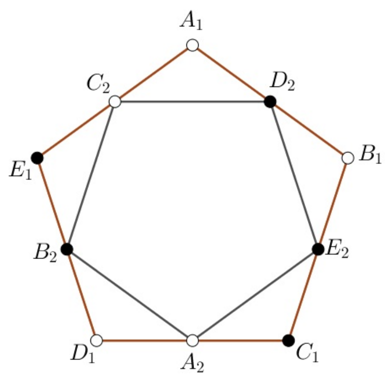
In this case also, there is a monochromatic cyclic trapezium. Can we reduce the number of pentagons further? Again, the computer found no counter examples for three pentagons. The proof used in the case of four pentagons deduced the following fact:If the first row is 00011, then the other rows can only be one of

10110,  01110,  11001

If we choose two among these, then it is easy to see that the corresponding pentagons contain an isosceles trapezium (using the pairs of parallel lines enumerated earlier). Similar argument for the case 01010. Thus three pentagons are enough to draw the conclusion. On further thought, the original argument used in the case of 11 pentagons can be modified to prove the result for three pentagons!



Let us call a line good if both its ends have the same color. In whatever way we color a pentagon, there are at least four good lines. In three pentagons, there are 12 good lines. The number of possible directions is 5 and hence by Pigeonhole Principle, there are three lines in the same directions. These are parallel and since their ends have the same color, we can choose two of them to form an isosceles trapezium with all vertices having the same color. Obviously, two pentagons can be colored such that they contain no isosceles trapezium (see the figure below).




It is doubtful whether the result that the minimal number of pentagons is 3 could have been found without the exploratory path taken using the computer. Also, the naming convention (used by one of the students in his solution) led to the unexpected mapping to a known problem of monochromatic rectangles in rectangular grids colored with two colors. The anticlimax of finding the shortest proof for the strongest result is a good illustration of how an elegant solution to a problem is discovered. This Aha moment is the fascinating part of problem solving.

The author would like to thank C R Pranesachar for the elegant proof of the three-pentagon case and Sahil Mhaskar for several interesting discussions.

# 8 How to handle mathematics lectures online using technological tools by Surendranath Reddy

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Abstract

Online teaching has been adopted all over the world to continue with teaching during the COVID-19 pandemic. But handling mathematics lectures virtually is not easy unless we use handy technological tools. In this article, we discuss some useful tools to handle mathematics lectures online with ease. We explain in detail how to use Microsoft OneNote for digital note taking with the help of pen-tablet, Microsoft forms and Testmoz for online exams, Edmodo as a Learning Management System and Open Broadcaster Software (OBS) for recording lectures, live streaming, etc.

## 8.1 Introduction

Due to COVID-19, all education institutions have been closed for physical classes since March 2020. Due to the lock-down, E- learning through various digital platforms has gone up exponentially. Schools and colleges started adopting online teaching even though their teachers did not have proper training to use the technological tools required for online teaching. As a result students as well as teachers faced many difficulties. To overcome these difficulties, Swami Ramanand Teerth Marathwada University, conducted three online faculty development programs and trained more than 17000 teachers across the country on ICT tools for online teaching, learning and evaluation. Some of the tools demonstrated in those programs have been discussed here, like OneNote, Microsoft Forms, Testmoz, Edmodo as LMS and OBS Studio for recording and live streaming.

This article has emerged from a lecture delivered in a workshop session during the second (online) annual conference of the MATHEMATICS TEACHERS' ASSOCIATION (INDIA) held during 5-6 September 2020. The lecture was titled Digital technology and online resources: A means to support mathematics learning during the Covid-19 pandemic.

## 8.2 Online Mode of Teaching, Learning, and Evaluation

Similar to the physical classroom learning, online mode also has a structure, namely, content preparation, delivery, students' progress and evaluation. Below are some important points to consider during online delivery of lectures.

- It is better to prepare content for Online Lecture in the form of presentation or digital notes taking.

- There are different digital Platforms available to deliver lecture like Cisco WebEx, Zoom, Google Meet, Microsoft Teams, OBS.

- To track the students' progress, we can use Learning Management Systems such as Edmodo, Google Classroom, Microsoft Teams, Moodle.

- Various tools for conducting online exams for students' evaluation includes Microsoft forms, Google forms, Testmoz, etc.

- Proper use of YouTube can make online teaching more effective as it has many useful videos from which one can learn about the platforms and methods given above.

## 8.3 Microsoft OneNote for digital note taking

In a physical classroom, teachers enjoy explaining steps and concepts using blackboard or a whiteboard. Even in virtual platform, there are several tools which can be used as virtual whiteboards. In fact they can be used for disseminating many other useful information than just writing. One of the most useful tools for digital note taking is Microsoft OneNote. It comes inbuilt and freely with systems using Windows 10. Systems operating on earlier versions of windows can also download Microsoft OneNote freely from their official website. Some of the features of OneNote are listed below.

- OneNote is often described as a digital 3-ring binder. It has Notebooks, Sections, and Pages available.

- Pen-tablets like XP-pen, Wacom or Iball allow one to write or draw on a page opened in OneNote.

- Documents are automatically synchronised to OneDrive and can be accessed from a laptop, tablet, or mobile phone connected to the internet.

- There is facility to annotate pdfs easily, insert images, video etc.
- One can search text even from hand written notes, images, etc.
- It is easy to share and collaborate using documents on OneNote.

To access OneNote, one needs to login with an Institutional Microsoft account or free Microsoft outlook account. More details can be found in the following video. YouTube video: https://youtu.be/PzgyQ0WpigQ?t=4518 .

## 8.4 Microsoft Forms

One of the most challenging tasks is to conduct online exams. Some of the tools for conducting exams are Google Forms, Microsoft Forms, Testmoz, MOODLE, etc. Microsoft Forms have some advantages over Google forms as Microsoft forms support equation editor and LaTeX typesetting. So it is very handy for mathematics teachers to type mathematics questions. Microsoft forms also have the option to branch questions and shuffle questions with a locking feature. This locking of some questions is essential as the name of the student and roll number should not be shuffled. The best part is that the analysis of the exam can be shared with students or with administration via a clickable link. Themes offered by Microsoft forms feel refreshing and one can customise the form and create themes. The following links can be references for a demonstration of Microsoft forms to conduct online assessments.

YouTube video: https://youtu.be/ljjqCij75AY

Presentation (pdf): https://tinyurl.com/MS-forms-Surendra

## 8.5 Testmoz

As discussed above, Microsoft Forms have several advantages for conducting exams. However, a major disadvantage with Microsoft Forms is that the form will not be submitted automatically after the time set for the assessment is over. So late submissions are not accepted automatically and accessing them is therefore a problem. To overcome this problem, one can look into another useful tool called Testmoz. It is a very fast and simple way to generate an online exam for students. Testmoz is a browser based test and can be accessed by the students without any registration and works even at a low bandwidth. Testmoz is designed to be mobile friendly. Testmoz has free as well as paid versions.

Free version has limitations like not more than 50 questions and not more than 100 results per test. It has no mathematics support or question banks (pools, etc. whereas the paid version (which cost approximately Rs 2000/-) has access to all the tests at one place, with no limit on the number of questions and or on the analysis of students' performance. The best feature is that by using the pool option, we can set n random questions out of m questions from that pool. For example, we can randomly set 5 questions out of 10 questions to each student. The paid version also supports question upload and importing questions from previous tests. Testmoz also supports browser restrictions like disabling of right click, copy, paste, etc. There is no upload option as of now, which is required for students to upload their answer as an attachment and one hopes that they may add this feature soon. For complete demonstration of Testmoz, one can refer the following links:

YouTube Video: https://youtu.be/gdAX25GA3f4

Presentation (pdf): https://tinyurl.com/testmoz-surendra

## 8.6 Edmodo- Learning Management System

A Learning Management System (LMS) is necessary to connect with students and track the progress of the students through assignments, quizzes, etc. There are several LMS. Some of them are free and others are paid. Google Classroom, Edmodo and MOODLE are some of the free LMS. MOODLE is good if the institution has its own server, otherwise there are some limitations. Google Classroom requires an institution to sign-up on G-Suite. So we recommend the use of Edmodo, which is far better than Google Classroom in our opinion. Edmodo can bring students, teachers and parents in contact on the same platform. It offers a social learning platform with a communication stream just like Facebook. Students can be awarded badges based on their performance and even teachers can earn badges by means of reaching different training levels. Edmodo offers poll options, assignments and quizzes with math support. Through auto grading and analysis, one can trace the progress of each student easily. Edmodo can also be integrated with Microsoft OneDrive for an easier way of sharing the content. For a demonstration on how to install and use Edmodo, please refer to the following.

YouTube video: https://youtu.be/sy1li92J_3I

Presentation (pdf): https://tinyurl.com/edmodo-rupali

## 8.7 OBS Studio

This is by far the best software I came across during the lockdown. Open Broadcaster Software (OBS) Studio, is a free and open source software. It is extremely useful for teachers in preparing online lectures and live streams. OBS Studio can be used for recording our computer/laptop screen and can live stream on various platforms like YouTube, Facebook etc. OBS is mainly divided into different scenes and each scene contains one or more sources put together. Some of the useful sources are display capture, video capture (webcam), audio input, images, text, etc. Each source has many useful filters. Using chroma key filter for video capture, we can remove the background of our webcam video. Similarly using noise suppression filter for audio, we can reduce unnecessary noise. OBS offers shortcut keys for each and every action which makes more easier to operate and make our video more professional. OBS is very useful for teachers as we can record our presentations in high quality and share with students for any time access. Using OBS, we can also give live lectures on platforms like YouTube, Facebook, etc where students can ask their doubts through chat box. OBS is certainly a boon for the teaching community. A demonstration can be found here:

YouTube Video: https://youtu.be/1hnmwePfNcg

Presentation (pdf): https://tinyurl.com/obs-surendra

## 8.8 Links for useful tools

Some of the useful software links have been provided for a quick download. We have also provided our YouTube video links for more ICT tools and their demonstrations.

| Name of the tool | Clickable link |
|---|---|
| OneNote | https://www.onenote.com/download |
| P3X OneNote on Linux | https://github.com/patrikx3/onenote |
| OBS Windows 64 bit | https://tinyurl.com/OBS-64bit |
| OBS Windows 32 bit | https://tinyurl.com/OBS-32bit |
| LOOM for recording | https://www.loom.com/desktop |
| Free recording | https://www.flashbackrecorder.com/express/ |
| HandBrake | https://handbrake.fr/downloads.php |
| YouTube Channel | https://tinyurl.com/surendra-youtube-channel |

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# 9 Crisis, Nation State and Education by Tathagata Sengupta

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Whenever humanity is faced with severe existential crises such as catastrophic wars, natural disasters, or deadly global pandemics, our common sense makes us expect that 'society' as a whole would come together to deal with the situation. Mainly because such crises affect everyone to lesser or greater extent, and require collective action in response.

## 9.1 Crises and Technologies of Governance

But history tells us otherwise. While on the one hand global pandemics have led to scientific developments in terms of new medicinal knowledge, and health and hygiene practices, they have at the same time also been major flash points when totalitarian governance of people by the authorities have been institutionalized. The Black Plague, or the Bubonic Plague, that had wrecked medieval Europe and North Africa for 4 centuries beginning from the 14th Century, is mainly known for the havoc it caused in terms of the hundreds of millions of lives lost. According to some estimates, the first round of the plague wiped out around 60% of Europe's population.[1]

However much less is talked about legislations that were passed by the ruling elites in the context of the plague, to keep the laboring classes in check. Rampant deaths for such a long period led to serious shortage of labor, causing an inflation in the cost of human labor. In response to this, the British Parliament passed the 'Statute of Labourers' in 1351 that "employees [who were idle and [...] not willing to take employment [...] unless for outrageous wages], both men and women, should be obliged to take employment for the salary and wages accustomed to be paid in the place where they were working in the 20th year of the king's reign [1346], or five or six years earlier; and that if the same employees refused to accept employment in such a manner they should be punished by imprisonment." [2] The pre-Plague wage rates it refers to were the abysmally low wages that were in place before the shortage of labor had occurred. In 1363 a Sumptuary Law[3] was brought which decreed the quality and colour of cloth that common people at different levels of society (below the nobility) should wear, and restricted the common diet to minimal basics. This was presumably to keep in check upward cultural mobilities of working sections of society who had - at least the ones who survived the plague - benefited from somewhat inflated wages and control over orphaned land. In our own present times, we witnessed how legal tools like the 'Essential Services Management Work' are being used to force the working classes to conduct highly risky and deadly jobs such as sanitation work and other so-called 'essential services', under the threat of police action and absolute poverty. While the hegemonic classes 'work from their homes'. Crisis-induced creation of such legal and moral technologies for tightening control over the working classes and patriarchal and colonial subjects, have been the basic cloth out of which the fabric of modern institutions of governance have been cut out. It includes nation states, municipal governments, quarantines, vaccinations, housing reforms, sanitations, maternity norms - and notably - educational reforms.

All such institutions of modern governance have had dual faces - convenience and empowerment for the tiny hegemonic classes on the one hand, and ruthless control over the massive working classes and patriarchal and colonial subjects on the other. Around 1650, conditions of living in Europe began to improve drastically for the hegemonic classes. Thomas Bollyky in his book titled 'Plagues and the Paradox of Progress' [4] documents the role played in such improvements by social and political reforms such as shifting to the suburbs from the urban centers thus reducing over-crowding, state institutions and public health systems, private philanthropy, etc - much before vaccines and other medical advances were even conceived of.

The working classes were intended to be instrumental beneficiaries of such measures primarily from the concerns of reinforcing in them confidence in the ruling apparatus, keeping them healthy so that their infections don't spread to the elites, and keeping them healthy and reproductive to ensure a steady supply of labor. "Greater appreciation of population growth as a means of expanding foreign empires, armies, and national economic growth led some European governments to invest in infant survival and motherhood education programs as 'matter[s] of Imperial importance'," points out Bollyky. Dorothy Porter in her book titled 'Health, Civilization, and the State: A History of Public Health from Ancient to Modern Times' [5] describes the so-called 19th Century 'pauper schools' - schools for the poor - whose purpose was mainly to train the poor "to 'discharge their social duties' through training in 'correct social habits' as much as providing workers with technical skills." Citing 20th Century eugenicists, Porter reminds how "Sex educationalists in the social hygiene movement believed that 'eugenics will destroy that sentimentalism which leads a woman deliberately to marry a man who is absolutely unworthy of her and can only bring disease, degradation and death'." Concerns over declining population levels at the end of the First World War put pro-natalism and infant and maternal welfare high on the political agenda. "A massive public education campaign encouraged its demobbed sons to procreate as fast and as often as possible," writes Porter. One can go on and on citing similar examples across all critical epochs of human history. But in short, the historical relationship of hegemonic social and political institutions - such as establishment education programs - with the common working population, has been one of instrumental negotiation of crises while maintaining the power status quo, with the hegemonic classes as the ultimate intended beneficiaries.

## 9.2 Education policies and making of the Indian Nation State

Let us now turn briefly to the history of Indian education system, before we get to the present times. The Indian Constituent Assembly proposals on Education were based on several earlier policy documents drafted by different political movements - documents that have been broadly categorised as ''Historical Constitutions" . These documents formed the foundations of the Constitution making process. We mention here three broad categories of these 'Historical Constitutions'- the ones by the Indian nationalists (mainly led by the Indian National Congress), ones by the Socialists, and the ones by the Anti-Caste Movements. The Nationalist documents include the 'Constitution of India Bill' of 1895 [6] (also called the Swaraj Bill), the 'Karachi Resolution' of 1931 [7], the 'Commonwealth of India Bill' of 1925 [8], and the 'Nehru Report' of 1928 [9]. A key example of the Socialist proposal was the 'Constitution of Free India' of 1944 [10]. Proposals put forward by the anti-caste movement include the 'Poona Pact' of 1933 [11], 'Political Demands of the Scheduled Castes' of 1944 [12], and 'States and Minorities' of 1945 [13].

Here is what the Nationalists propose regarding the right to education in the documents listed above: "State Education shall be Free in the Empire", "Primary Education shall be Compulsory in the Empire", "Free Primary education", "right to elementary education", "right to free elementary education". The Commonwealth of India Bill in fact put limits on electoral rights based on income, land ownership, literacy, and educational qualifications. In summary, the idea of mass public education for the 'upper' caste-led Indian Nationalist movement was basically 'free and compulsory primary education'. The only real way in which the Socialist proposal differed from this was the introduction of the adjective 'secular'. "Education for all children up to the age of fourteen shall be free, compulsory and secular," said the 'Constitution of Free India' drafted by M.N. Roy.

Now let us contrast the above with the proposals drafted by the Anti-Caste Movements. "An adequate sum shall be ear-marked for providing educational facilities to the members of Depressed Classes", said the Poona Pact - an agreement between Dr. B.R. Ambedkar and M.K. Gandhi on the political representation of the Depressed Classes. The document titled 'Political Demands of Scheduled Castes' formulated by the Scheduled Castes Federation - the national body spearheading the Dalit movement in the subcontinent at the time, declared "no Constitution shall be acceptable to the Scheduled Castes unless ... it contains within itself provisions for securing the following purposes: (1) For earmarking a definite sum in the Budgets of the Provincial and Central Governments for the Secondary University and Advanced Education of the Scheduled Castes " Describing the 'spread of higher and advanced education among the Scheduled Castes'as of 'vital importance', the Working Committee of the SC Federation demanded that "the Constitution should impose an obligation upon the Provincial Governments and the Central Government to set apart adequate sums, as may be specified by the Constitution, exclusively for advanced education of the Scheduled Castes in their annual budgets and to accept such provisions as a first charge on their revenues." In a radical inversion of the majoritarian logic of constraining legislative rights in accordance to educational qualifications, Dr. B.R. Ambedkar in his manifesto titled 'States and Minorities'demanded that the weightage of legislative representation carved out from the share of the majority "shall be divided among all minority communities equally or in inverse proportion to their (1) Economic position, (2) Social status, and (3) Educational advance"(Emphasis added). In the section on 'Special Responsibilities', the manifesto says, "Governments — Union and State — shall be required to assume financial responsibility for the Scheduled Castes and shall be required to make adequate provisions in their budgets." The budget allocation for the purpose is to be "in proportion to the population of the Scheduled Castes to the total budget of the States." It also demands that "the responsibility for finding money for foreign education of the Scheduled Castes shall be the responsibility of the Union Government." All this is in addition to the provisions of free primary education that is also the responsibility of the State. For Dr. Ambedkar, the provision for free and compulsory education was not for the manufacture of 'trained workers', but in fact the opposite - "If the child is not to be employed below the age of 14, the child must be kept occupied in some educational institution... that is why I say the word ‘primary’is quite inappropriate in that particular clause”, he said in the Constituent Assembly discussion. His view of education was not to produce future working bodies, but to protect and nurture the human spirit and it’s precious creativities.

What was the eventual outcome of the Constitutional vision of education of the newly born nation state? The Sub-Committee on Fundamental Rights with members including Dr. Ambedkar, Sardar Harnam Singh and K.M. Munshi, had formulated substantive provisions on education as a universal right or entitlement. The SC Federation’s demands of State responsibility for Advanced Education were already muted at this Sub-Committee stage of ‘nation building’. But still the final provision at least retained the characterization of free primary education as a fundamental right. “Every citizen is entitled as of right to primary education and it shall be the duty of the State to provide within a period of ten years from the commencement of this Constitution for free and compulsory education for all children until they complete the age of fourteen years” - the Sub-Committee proposal read. In April 1947, the Advisory Committee which sat to consider the Sub-Committee’s report, seemed startled at the proposal of free primary education as a fundamental right, and asked “Is this a justiciable right? Supposing the government have no money?”. One of the members, Alladi Krishnaswamy, demanded: “I want the deletion of this clause”. Sure enough, the Committee promptly dispatched the provision to the non-legally enforceable Directive Principles of State Policy.[14] On 23 November 1948 the Constituent Assembly took up the debate on the Draft Article on Education. By now it was already relegated to the Directive Principles and had lost all meaning. But still the Committee passed a further amendment to replace the phrase “Every citizen is entitled . . .” to “State shall strive to . . . ”.[15] And that’s that.

## 9.3 “Broken Slates and Blank Screens”

The compounded effect of this over the decades can be gleaned from the recently published report titled ‘Broken Slates and Blank Screens - Education under a Lockdown’, by the People’s Union for Civil Liberties (Maharashtra).[16] Before specifically delving into the question of ‘digital divide in education’ exacerbated by the pandemic lockdown, the report talks about the already dysfunctional state of education in the country - focusing particularly on Maharashtra. The report holds “the failure of the government to provide the continuum for schooling” from the primary to secondary levels as a key reason for catastrophic pushing out of students from formal education. "Total number of government schools in the State offering Primary to Higher Secondary education are merely 339 as against 4,885 private schools in this category, and the total number of government schools with Primary to Secondary transition are 841 as against 8,260 private schools," the report notes. Surely this says something about the validity and farsightedness of the Anti-Caste Movements' concerns about Education at the eve of nation formation. "For children from marginalised sections the possibility of continuation to Upper primary and Secondary schools is not at all assured as here too the availability of government schools providing (subsidised schooling) is seriously poor i.e., government schools with Upper Primary and Secondary are a mere 328 as against private schools in this category being 5,851" - the report further adds.

The other key aspect that the report highlights, is the treatment of education "as a privilege or a commodity rather than a fundamental right." We have already noted how the possibility of education as a fundamental right - the centuries-old demand of anti-caste reformers like Jotiba and Savitribai Phule, Fatima Sheikh, Pandita Ramabai, Guruchand Thakur, Dr. Ambedkar, Rokeya Sakhawat Hossain and many others - was murdered in the very womb itself during the Constitution drafting process. Though a recent attempt was made by sections of the civil society in framing the Right to Education Act, it is a rather weak piece of legislation and it's relevance in addressing the structural issues of the Indian education system has been called into question by many quarters. Arguably one of the main positive provisions of the Act - the Non-Detention Policy - has also been diluted through recent amendments, weakening it even further. Many have argued in fact that the RtE has mainly aided the ongoing 'commercialization of education'through legitimizing the replacement of the State's moral responsibility with so-called Public-Private 'Partnerships'(a euphemism for something that should be more accurately termed as a Parasitic relation based on public costs and private profits). That is a whole different discussion, but for the present context this brings us to the question of 'education as a privilege and a commodity', and in particular, the ongoing 'digitization of education' particularly turbo-charged by the pandemic crisis.

## 9.4 'Quality Education', 'Merit' and the Indian 'General Categories'

For a long time now there has been a push towards the digitization of education - coincidental with the digitization of economy (what has been called the 'platform economy') and of governance (what should be similarly called 'platform governance'). The irony of (digital) commodification of knowledge and education is quite clear - both the internet technology and knowledge in general are productions of the public sector. The internet was invented and advanced purely through State-funded (which means money from the citizens) Universities and research institutions in the 'first world'[17]. Knowledge is of course by definition, a 'commons' - produced, nurtured and propagated by global society as a whole. It is the irony of our age that a few digital corporations are allowed to establish their business control over these two public domains that are 'public by definition' and ought to be commonly owned and distributed.

Digitization of education in India marks a new advanced stage of a historic dual phenomena that were kicked off in the 1980s. Talking about the history of the Indian Middle Class, Sanjay Joshi notes, "the period from 1980s and 1990s to the present appears to be qualitatively different from the near- hegemonic position occupied by middle class of the Nehru era . . . Three landmark events at the start of the 1990s highlight these transformations—one, the agitation against the recommendations of the Mandal Commission (1990); second, policies of economic liberalization adopted by the Government of India (1991); and third, the December 1992 destruction of a 16th-century mosque by a Hindu nationalist mob that wanted to build in its place a Hindu Mandir (temple)." [18] The 'dual phenomena' are thereby - economic liberalization and cultural patriarchal nationalism. The Mandal Commission reforms were a civilizing moment for the Indian hegemonic institutions, particularly the educational institutions. For the first time it led to the mass entry (although to a far lower extent than what ought to have happened even if one goes by the simple logic of proportions, owing to various ingenious ruling class techniques of sabotaging the policy of reservations) of people from Dalit-Bahujan communities into the the academic institutions that were prior to this nothing less than upper caste fortresses. Of course the 'General Category' (read 'upper caste') Indian Middle Class didn't take this cheerfully. Like Joshi recounts, "there was outrage from the middle class in particular. The largely upper-caste groups took to the streets in protest in unprecedented numbers, and peers in the intelligentsia, as well as the print and television media, championed their cause. Shrill demands in support of 'meritocracy'(oblivious to the overwhelming majority of upper-caste students and job holders) were highlighted by dramatic self-immolations of upper-caste students. It was a movement to defend the privileges the upper-caste middle class had taken for granted."

On the one hand the threat perceived by the upper castes from the autonomous empowerment of Dalit-Bahujan communities led them to galvanize Hindu supremacist cultural politics (symbolized and led by the Ram Janmabhumi movement), one crucial aspect of which is to re-establish control over quality education and knowledge itself. On the other, this “coincided with an economic regime that favored economic liberalization, a period when an older system that strictly regulated private enterprise through laws and licenses gave way to one that was more favorable to both domestic and international capital.” Sure enough, the education industry was no exception to this, and was in fact one of the first sectors to undergo rampant privatization. This attempt, strictly speaking, was not a product of the ‘90s, but had in fact begun in the early ’80s itself. The first round of liberalization was engineered by Dr. Manmohan Singh in 1985 as the Head of the Planning Commission under the Rajiv Gandhi Government that was faced with a Balance of Payment crisis. The same Rajiv Gandhi Government had brought in the National Policy on Education in 1986 - precursor to the recent New Education Policy of the Modi Government. In 1985, the Ministry of Education was renamed into the Ministry of Human Resource Development - sealing state approval on the growing middle class consensus of ‘Humans’as mere repositories of ‘Human Resource’(which is basically a fancy term for ‘labor’and ‘statist morality’under a patriarchal capitalist nation state), and ‘Education’as ‘Human Resource Development’. Of course this kind of philosophy coincides with the other emergent consensus that nature is a repository of ‘natural resource’which is deemed valueless unless ‘economically tapped’. Accordingly the NPE of 1986 defines Education as something that “develops man-power for different levels of national economy”, and something that has “an acculturating role. It refines sensitivities and perceptions that contribute to national cohesion, a scientific temper and independence of mind and spirit.” Even though phrases like ‘scientific temper’and ‘independence of mind and spirit’were thrown in, the drafters of the policy didn’t seem very concerned about the possibility of a basic logical problem - “What if the idea of ‘national cohesion’went against the idea of ‘scientific temper’or the ‘independence of mind and spirit’?”The reason why this was not even a concern is that “acculturation towards national cohesion”is indeed the ‘real thing’, and things like scientific temper, etc., are allowed only as long as they are subservient to such acculturation.

## 9.5 Likely End of the Short-Lived Indian Public Education

Anyhow, post ‘80s, and particularly since the ‘90s, the Indian middle class - largely upper-caste in it’s composition - took off on the high jets of private education, true to their cultural consensus of a liberalized economy on the one hand, and a project of supremacist acculturation on the other. “Gradually the middle-class abandoned state-run schools – preferring private/aided ones providing seamless schooling from class I to X,” notes the PUCL report. The vast majority of working Dalit-Bahujan-Adivasi classes of the country were left only with ‘free primary education’in totally sabotaged State-schools, and things like ‘Non-Formal education’, ‘vocational education’, etc, etc. Faced with the lack of continuum from primary to secondary levels of education, no reservation policy till Class XII, scrapping of the ‘No Detention Policy’, and a host of other structural factors, Dalit-Bahujan-Adivasi students - particularly women students from such communities - have been forced to leave the space of education in a structured manner across generations. “The latest Economic Survey of the state shows that more than half the children enrolled in primary schools do not reach class X,” notes the PUCL report. As students have been leaving the schools, the Government has been using this as a prompt excuse to ‘rationalize’ schools - which means shutting down of schools that don’t have enough number of students enrolled. More than 13000 schools were shut down by the Maharashtra Government 3 years ago using this logic.[19] This is no different from chopping off body parts because one’s own lifestyle decisions have led to infections in those body parts.

All this, together with reduction in proportional budget allocation for State-run schools, under-spending, and simultaneous increase in Government subsidies and grants to private and PPP schools - have meant only one thing: Education has been taken away structurally from the people at large, and captured by the hegemonic classes through gate-keeping, sabotage and monetization. The Indian Public Education – that took shape only post 1947 - is now looking poised for an abrupt and early childhood death. NSSO data itself establishes the economic capture on firm grounds - noting that the richest 5% in urban India spend 29 times more on education than households in the middle of the rural income distribution.[20] The hegemonic capture is clear from the social composition of higher Indian academia, where 95% of all Central University Professors, 93% of all Associate Professors, and 66% of all Assistant Professors are from the hegemonic ‘general’castes, although the population percentage of these castes combined is no more than 30%. In particular the posts of Professors and Associate Professors have allowed 0% OBC and less than 1% ST and only around 5% SC appointees - after 70 years of ‘nation building’.[21]

The ongoing polarisation of the quality and access of education also fits neatly with the corresponding polarisation of the landscape of professions between a small number of high-paying cognitive jobs and a huge ocean of depressed labor-intensive manual jobs, as more and more routine middle-income jobs get mechanized. But that is a story for another time.

## 9.6 The Digital Education Commodity

This brings us to the specific case of COVID-induced digitization of education. The 'digital divide'is a known reality and has been much discussed, so we won't get into those same details here. It suffices to just say here is as some have been pointing out, what was possibly a 'digital divide'so far is now a 'digital partition'. Talking about the school closures due to the lockdown, the PUCL report notes, "To take an example from Mumbai, according to a survey by the Education Department, Mumbai Municipal Corporation, about 28.12% children from class 1 to 8 had left with their parents joining the exodus ... The estimate is that about 30% students were not coming back. Online enrolment for current academic year is reduced by half." About access to digital equipments, it says, "Out of about 2 lakh 14 thousand students in elementary schools only about 47.78% students have access to smart phones and 46.74% have remained connected to studies through various online learning initiatives of the department and the state government." Reports of students from humble backgrounds killing themselves out of the stress of non-access to digital education have become a regular occurrence. "The crisis and labour market shock can push millions of vulnerable children into child labour. Already, there are an estimated 152 million children in child labour, 72 million of which are in hazardous work," notes a recent ILO report.[22] The lockdown has made this far worse. The PUCL report quotes Manish Sharma of Bachpan Bachao Andolan (BBA) as saying "We have sure-shot information coming from the field that human traffickers have begun contacting and extending support to the families migrating back home so that they could use their children as bonded labour in the time to come." This is a good opportunity to remind ourselves of Dr. Ambedkar's understanding of education in relation to child labor.

Anyhow, meanwhile a literal scramble is taking place among newly mushroomed education corporations to claim chunks of the lucrative pie of the new digital education market. No less than billions of dollars are being invested by global corporations into this market, through 'ed-tech companies' like Byju's, UnAcademy, etc.[23] And the establishment academia is busy investing it's energies and worries into ensuring that the experience of online education gets to become as smooth and rich as possible for that super-thin layer of studentclients who have access to it. But in effect, even for these sections, education - decoupled from the social and material reality - is being reduced to pedagogic artefacts and commodities such as video lectures, online tutorials, interactive modules. In turn, this emerging world of 'education artefacts'is being quickly monetized through highly volatile global markets. But there is one thing that market systems don't want to entertain - criticality. Or at least, the kind of criticality that can not itself be marketized. Thus what we are seeing is also the end of possibilities of any form of 'critical education', or in simpler words, 'education'.

There are credible apprehensions that this shift towards a digitized, bipolar, totally torn education system is going to stay and build upon itself even after the pandemic crisis is over. The most relevant reason for this belief are the billions of dollars that are currently being invested into 'ed tech', the extensive market surveys that are being conducted, the shares that are being sold, the loans that are being raised, the bets that are being made. No such large-scale investment is going to quietly go down ever, and the move towards institutionalization of such monetized technologies of market-driven education seem like a sure eventuality as of now.

## 9.7 What is to be done now?

The question for us then is, what is it that should be done now?

The fundamental answer to this can only be large scale public consultations across all communities, specifically led by oppressed community organisations and mass movements, teachers' unions, students' unions, feminist movements and other bodies that have been struggling for the right to education as a public good, and for knowledge as a commons. This can only happen through large scale organizing, as can be witnessed across large parts of the world today - such as in Chile, South Africa, Ethiopia, Kenya, Turkey, US, and elsewhere.[24] We need to connect the struggle for education with other social movements such as those against Fascism, against capitalism, against caste, against patriarchy, against the militarized nation state's control over the commons such as the struggles for 'Jal Jangal Zameen' across different parts of our country. We need to learn from experiences of progressive education systems such as those from Mexico, Cuba, Syria, Brazil and elsewhere. We need to learn about the histories of struggles for quality education in the subcontinent itself, mainly led by the various Anti-Caste and Anti-Supremacist movements. We need to talk about the questions of education faced by the students in Kashmir - a region that has been forcefully kept under a brutal militarized lockdown for more than a year now in the name of 'security'and 'national cohesion'(this ought to remind us of the NPE 1986 'visions').

What is broadly clear is that at the bare minimum, any such organizing has to be built on the basis of serious engaging debates, discussions and studies of all possible dimensions of education and knowledge:

1. Political-Economic Dimensions: Involving the questions of State responsibility of free and continuous advanced education for all, based on the premise of 'reparations'and 'redistribution'of the rightful share of material resources for education that have long been captured from the vast majority of oppressed people of this part of the world. This must particularly include renewed struggles for reservations in the academia right from primary education all the way to the top research institutions such as TIFR, IISc, HBCSE, etc., increased public spending in Education, Anganwadi, ICDS, all the way up to public funded Colleges and Universities, and so on.

2. Cultural-Epistemological Dimensions: Involving questions of 'recognition'on the nature of what is considered legitimate knowledge and pedagogy, learning to cherish and value difference as the only possible site of knowledge exchange, and rejecting the idea of 'secret' or 'privately-owned' knowledge fashioned under caste and corporate patriarchy.

3. Historical-Ethical Dimensions: Involving questioning long-held assumptions by looking into the mirrors of history, re-examining the ethics of education and the ethical-epistemological-historical aspects of various disciplines like the Sciences, Social Sciences, and Mathematics.

4. Pedagogic Dimensions: Involving learning and teaching ethics and practices, practices of listening and forging dialogs, rejecting the cult of 'secret knowledge', and working towards a culture of education that is centered around learners' control over means of knowledge production, treating students as future colleagues, and rejection of 'guru-shishya' hierarchies.

5. Internationalist Dimensions: Involving forging collaborative dialogs and projects with other anti-hegemonic forces across the world faced with shared crises of humanity such as the crises of war, Fascism, Capitalism and Ecological catastrophe.

In addition we need to construct new systems of public-owned knowledge exchange such as the setting up of counter-platforms for free, decentralised, accessible and socially-accountable high-quality education, that helps forge autonomous horizontal knowledge exchange networks across oppressed and vulnerable communities without the mediation of the hegemonic classes and castes, setting up of major translation projects, working on subversive technologies, training students on subversive technologies, and building up of a rich 'Knowledge Commons'centered on the concerns of representation, recognition and redistribution. In summary we must work towards practically re-defining knowledge and education as fundamental instruments to counter all forms of power and control.

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# 10 Vertex Labeling of a Half-Cube to Induce Desired Face Labels: A Mathematics Exploration Activity for School/-College Students
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Abstract

Using digits $0, 1, 2, \ldots, 9$, label the ten vertices of a hexagonal half-cube so that the induced labels of six faces (other than the hexagon), given by the sums of their vertices, are three of a kind and three of another. Find all such labeling using mathematical reasoning.

## 10.1 Introduction

In the first issue of Blackboard [6], Ramanujam called on mathematics teachers to provide opportunities for every child to engage in open mathematical explorations; and hoped that MTA(I) would become a forum to share such activities. Following in this spirit, in the second issue of Blackboard [5], Jayasree offered an exploratory mathematical activity conducted for grade IX students in a Corporation school in Chennai. Continuing in this spirit, here I offer a mathematical puzzle that can be explored by students in secondary school to find one or more trial-and-error solution(s), by college students as a practice in writing mathematical proofs that they have identified the complete collection of all solutions, and by research mathematicians to pose and solve more exotic versions of the puzzle requiring advanced mathematical features.

Let me explain how the puzzle was born. My colleagues and I study random walks on the vertices of different graphs. Having studied the random walk on a hexahedron (commonly known as a cube), see [4], we turned our attention to half-cubes obtained by plane cuts through the center. We were pleased to note that there are exactly three half-cubes topologically distinct from the cube itself—rectangular, rhombic, and hexagonal—named after the new surface generated by the cut. We leave the proof of the above claim as an exercise to the reader. In particular, the hexagonal half-cube fascinated us the most because of its three-fold rotation and reflection symmetries. Let us take a moment to define it explicitly.

Definition 1. (Hexagonal half-cube): When a cube is cut by a plane passing through the center and orthogonal to a diagonal, we obtain two identical hexagonal half-cubes.

Figure 1 shows (a) a cube sliced in half (the diagram is taken from [4]), (b) a hexagonal half-cube, and (c) its planar graph with vertices labeled alphabetically $a$ through $j$. (The lower-case letter represents a number, and the corresponding upper-case letter names the vertex.) See [8] for a description of how to obtain the planar Figure 1(c) starting from the three-dimensional object in Figure 1(b). A hexagonal half-cube has ten vertices, 15 edges and seven faces—three (isosceles, right) triangles, three pentagons (when the above-mentioned isosceles, right triangle is cut off from a corner of a square face of the cube) and one (regular) hexagon generated by the plane cut.

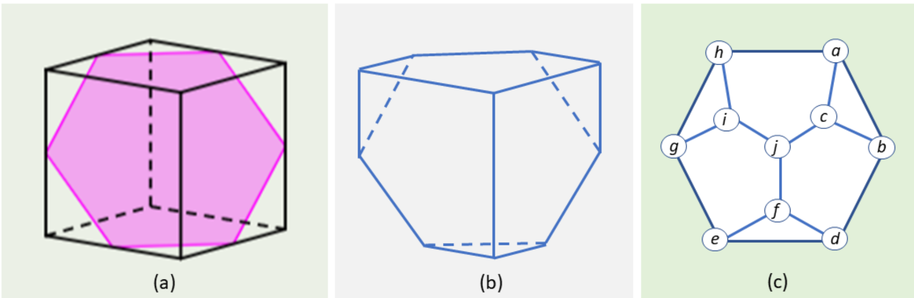


Figure 1: (a) A cube, (b) a half-cube, and (c) the planar graph of a half-cube labeled alphabetically.

Motivated by the geometrical symmetry of the three triangular and the three pentagonal faces in Figures 1(b) and 1(c), I solved the following curious puzzle in [8]:

> Puzzle 1: Label the ten vertices of the hexagonal half-cube with digits 0, 1, 2, . . . , 9 (using each digit only once) so that the induced labels of the three triangles given by the sum of their three vertices is a constant $T$ (the T-constant property), or the induced labels of the three pentagons given by the sum of their five vertices is a constant $P$ (the P-constant property), or both (the TP-constant property).

Previously, I had posed the TP-constant version of Puzzle 1 as a contest problem for the 2019 IUPUI High School Math Contest conducted over a six-week period. Of the fifty-some students who participated, about thirty students solved it using the trial-and-error method; but no one found the complete list of all TP-constant labeling. Only one student mentioned: "It will be nice to list all other solutions to this puzzle." On the other hand, a research mathematician in my department asked me whether the task was to prove that a TP-constant solution does not exist.

The paper [8] proves that, when not distinguishing labeling that are equivalent after rotation, reflection, or complementation (that is, subtraction from 9), there are 864 T-constant labeling, 544 P-constant labeling, and a unique TP-constant labeling. Each such labeling gives rise to 12 distinct labeling when allowing rotation, reflection, and complementation. In fact, six of the 12 TP-constant labeling yield ($T = 13,\ P = 25$), while their complements yield ($T = 14,\ P = 20$). Specifically, none of the TP-constant labeling exhibits $T = P$, proving that it is impossible for all six faces to receive the same induced label. My colleague welcomed this result involving impossibility.

Moreover, a TP-constant labeling also satisfies a D-constant property; that is, the sum of any pair of opposite vertices of the hexagonal face (that is, vertices on each of its three diameters) is yet another constant $D = 45 - T - P$. Solving an exercise posed in [8], we report that there are 34560 D-constant labeling. You may wish to prove this claim yourself before reading my proof towards the end of this paper. Moreover, a TD-constant labeling is also P-constant, and a PD-constant labeling is also T-constant. Hence, the 12 TP-constant labeling are the only TD-constant labeling and the only PD-constant labeling. Therefore, we may call them TPD-constant labeling.

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The solution to one mathematics problem is often the beginning of another problem. Jayasree [5] quotes from [1]:

> "One sure-fire way of generating a mathematical exploration is to ask the questions 'What if?' or 'What if not?' (Brown & Walter, 2005)."

We paid heed to this advice and asked: "What if the three faces receiving the same label are not necessarily the three triangles or the three pentagons?" What if one set of three faces receiving the same label consists of two triangles and one pentagon and the other set consists of the remaining triangle and two pentagons? Pursuing this question, we extend Puzzle 1, to allow the six faces (other than the hexagon) to receive induced labels that are three of a kind and three of another.

> Puzzle 2: Label the ten vertices of the hexagonal half-cube with digits 0, 1, 2, . . . , 9 (using each digit only once) in such a way that the induced labels of the three triangles and the three pentagons are two distinct numbers with frequencies three each.

At the outset we must explicitly state that finding a solution via a complete search of all $10! = 3,628,800$ permutations of digits 0 through 9 is expressly prohibited. One must use mathematical reasoning to find a solution. For a solution is nice; but the logic that leads to the solution is far more precious!

As depicted in Figure 2, Puzzle 2 is a collection of three puzzles of which the first one is Puzzle 1:

P2.1 The three triangles receive label $T$, the three pentagons receive $P$. Starting with a triangle and going clockwise the face labels are $(T,\ P,\ T,\ P,\ T,\ P)$.

P2.2 Two triangles and their intermediate pentagon, which is adjacent (sharing a common boundary) to both, receive label $K$, the remaining three faces receive label $L$. Again, going clockwise from a triangle the face labels are $(K,\ K,\ K,\ L,\ L,\ L)$.

P2.3 Two triangles and a pentagon adjacent to one of these triangles and non-adjacent to the other triangle receive label $M$, the remaining three faces receive label $N$. Thus, going clockwise from a triangle the face labels are $(M,\ N,\ M,\ M,\ N,\ N)$.

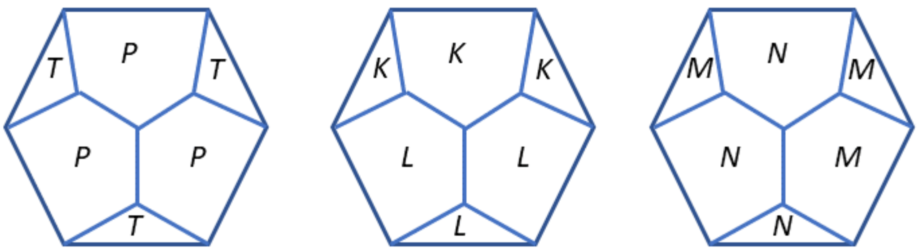


Figure 2: Induced face labels are three of a kind and three of another exhibiting three possible patterns.

For younger students just beginning to explore mathematics, and for adults pursuing other non-mathematical careers, a natural approach to solving Puzzle

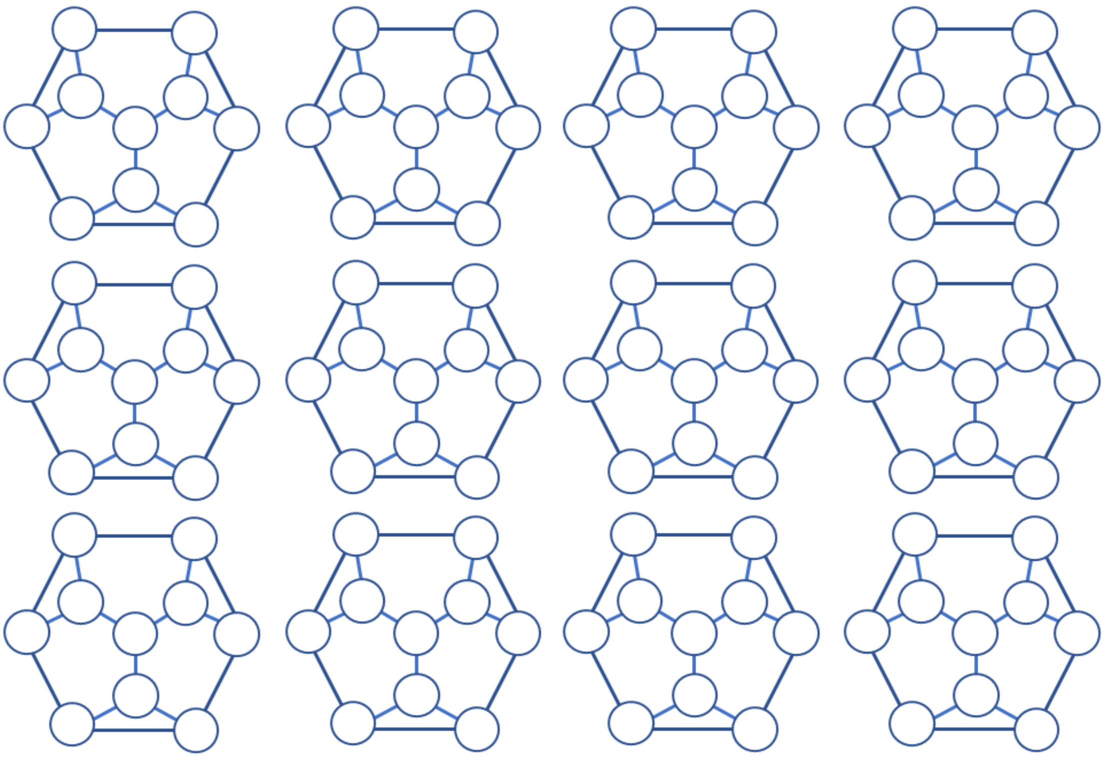

Figure 3: A template to label the vertices of a half-cube.

2 is to use the trial-and-error method. As an aid, we give them a template in Figure 3.

Admittedly, anyone trying to solve Puzzle 2 using the trial-and-error method risks becoming frustrated. But she may be sufficiently intrigued to keep on trying in hope of experiencing an ecstatic joy once she would find a solution. My dear wife, who did not study mathematics beyond secondary school, is one such candidate. I kept encouraging her that her chance of finding a solution is better than her chance of winning a state lottery. (I know this because she never plays the lottery.) Moreover, when someone wins the lottery, all other contestants' chances of winning drop to zero. (Or, if multiple winners are allowed, they must split the prize equally.) Not so when solving mathematics puzzles: Everyone who solves a puzzle—sooner or later—is a winner (of the ecstatic joy)! In any case, chances are high that a trial-and-error solver will consider the job done when she would find a solution.

However, to mathematics teachers (and their bright students) the real challenge is to list all labeling that solve each of the three variations of Puzzle 2. Specifically, to claim that there is no solution, the math enthusiast must prove the non-existence. Intentionally, the puzzle asks readers to label the vertices so that the induced face labels achieve a certain mathematical property, and lets the solver arrive at the more complete task of listing all labeling or proving that none exists.

Dear reader, the fact that you have read this far, allows me to encourage you to find out not just one solution, but all solutions to all variations of Puzzle 2 on your own or in collaboration with others before reading the paper further. Even after reading the paper you will find more puzzles to work on (or assign to your students).

## 10.2 Solutions

Let us begin with a new proof that it is impossible for all six faces of the half-cube to receive the same induced label—a proof independent of TP-constant labeling, which we alluded to above.

Lemma 2. All six faces of the hexagonal half-cube (other than the hexagon) cannot receive the same induced face label when the ten vertices are labeled with distinct digits 0 through 9.

Proof. The proof is by contradiction. Suppose that, if possible, all six faces receive the same induced label $Q$. We shall first show that such a labeling is D-constant. Comparing triangle ABC and pentagon ACJIH, we have $b = h+i+j$; and comparing triangle GHI and pentagon ACJIH, we have $g = a+c+j$. Hence, $b+g = h+i+j+c+a+j = Q+j$. Similarly, we prove that $d+h = Q+j$ and $a+e = Q+j$. Hence, the labeling is D-constant. Thus, the six vertices of the hexagon add up to $a+b+d+e+g+h = 3(Q+j)$. Adding to this sum the remaining four (interior) vertices, we obtain $3\,(Q+j)+c+f+i+j = 0+1+...+9 = 45$.

However, the vertices of the three triangles and the central vertex add to $3Q+j = 45$, which when subtracted from the previous equation yields $c+f+i+3j = 0$. However, this is a contradiction since $c,\ f,\ i,\ j$ are distinct digits between 0 and 9, implying that $c+f+i+3j \geq 6$. Therefore, all six face labels cannot be equal. □

Although Puzzle P2.1 (that is, Puzzle 1) has been already solved in [8], here we give a new and shorter proof. Finding new proofs to old results is another worthy pursuit for math enthusiasts.

### 10.2.1 Three Triangles Receive Equal Induced Labels and Three Pentagons Do the Same

Suppose that there is a TP-constant labeling $\pi$; that is, the three triangles receive induced label $T$ and the three pentagons receive $P$. Then adding the three triangular faces and the central vertex we have $3T + j = 45$, and adding all six faces we have $3T + 3P = 90 + c + f + i + j$. From the first equation, 3 divides $j$; and then from the second equation, 3 divides $c + f + i$. Also, any pair of opposite vertices of the hexagon add to $45 - T - P$; hence the labeling satisfies the D-constant property with $D = 45 - T - P$. Moreover, $9 - \pi$ is also a TP-constant labeling. Therefore, it suffices to consider $j = 9$ and 6. We study these two cases separately.

Case 1 ($j = 9$): In this case, we have

$$T = (45 - j)/3 = 12, \quad P = 21 + (c + f + i)/3, \quad D = 12 - (c + f + i)/3.$$

Since $j = 9$, the smallest value of $c+f+i$ is $0+1+2 = 3$ and the highest value is $8+7+6 = 21$, whence $5 \leq D \leq 11$. Next, let us form the triangle triplets to achieve $T = 12$. Since $j = 9$ is already used up and since all induced triangle labels must be 12, no two of digits $0, 1, 2$ can be around the same triangle. The triangle with a vertex labeled 0 must have its other two vertices labeled $(4, 8)$ or $(5, 7)$. Thereafter, splitting the remaining six digits into triplets that also sum to 12, we construct only two possible triangle triplets 048-156-237 and 057-138-246, each of which we designate as X-Y-Z (with each letter denoting the corresponding three-digit number). We will name the pairs of opposite vertices of the hexagon as XY, YZ and ZX, depending on which triangles are connected by the diameter, and check their sums.

For each triangle triplet X-Y-Z and for each permissible value $5 \leq D \leq 11$, in Table 1 we identify (at least) one pair of opposite vertices of the hexagon that cannot possibly sum to $D$.

Table 1: Opposite vertices of a hexagon that do not sum to $D$

| triplets | | | $D$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| X | Y | Z | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 048 | 156 | 237 | XZ | YZ | XY | XY | XZ | YZ | XY |
| 057 | 138 | 246 | XY | YZ | XY | XZ | XY | XZ | XY |

Hence, we have proved that there is no labeling with $j = 9$ that solves Puzzle 2.1. Let us not be disheartened; just move on to the next case. Perhaps we will find a solution, perhaps not. If Case 2 also fails to produce any solution, we will have proved the nonexistence of any solution. Either way we will win!

Case 2 ($j = 6$): In this case, we have

$$T = (45 - j)/3 = 13, \quad P = 19 + (c + f + i)/3, \quad D = 13 - (c + f + i)/3.$$

Since $j = 6$, we have $3 \leq c + f + i \leq 24$, whence $5 \leq D \leq 12$. Next, we form the triangle triplets that achieve $T = 13$. Again, no two of digits $0, 1, 2$ can be around the same triangle, for then that triangle label cannot be 13. The triangle with a vertex labeled 0 must have the other two vertices labeled $(4, 9)$ or $(5, 8)$. (Note that $(6, 7)$ is not possible, since $j = 6$.) Thereafter, splitting the remaining six digits into triplets that also sum to 13, we construct only two possible triangle triplets 049-157-238 and 058-139-247, each of which we designate as X-Y-Z, as in Case 1.

For each triangle triplet X-Y-Z and for each permissible value $5 \leq D \leq 12$, in Table 2 we document (at least) one pair of opposite vertices of the hexagon that cannot possibly sum to $D$. Only one combination allows all three pairs of opposite vertices of the hexagon—XY, YZ, ZX—to sum to $D = 7$, leaving open the possibility for a labeling that might solve Puzzle 2.1.

Table 2: Opposite vertices of a hexagon that do not sum to $D$

| triplets | | | $D$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| X | Y | Z | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 049 | 157 | 238 | XZ | XY | — | XY | XZ | XZ | YZ | YZ |
| 058 | 139 | 247 | XY | XZ | XY | XZ | YZ | XY | XZ | YZ |

Adopting another convention $c < f < i$, we write down (using the D-constant property) the unique TP-constant labeling with $j = 6$, $D = 7$ that solves Puzzle 1 as 571-328-049-6. Next, taking its complement (by subtracting from 9), we obtain another TP-constant labeling with $j = 9 - 6 = 3$, $T = 3 \times 9 - 13 = 14$, $P = 5 \times 9 - 25 = 20$ and $D = 2 \times 9 - 7 = 11$ as 428-671-950-3. These two solutions are depicted in Figure 4. These two solutions can be rotated and reflected to yield 6 labeling each, for a total of 12 labeling that solve Puzzle 2.1.

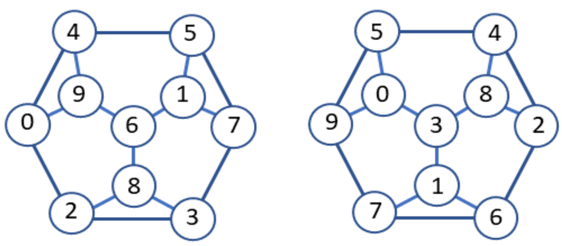


Figure 4: Each of these two TP-constant labeling, which are complements of each other, can be rotated and reflected to yield 6 TP-constant labeling, for a total of 12 TP-constant labeling.

### 10.2.2 Three Contiguous Faces Receive Equal Induced Labels, So Do the Other Three

Suppose that starting from a triangle and going clockwise the face labels are $(K,\ K,\ K,\ L,\ L,\ L)$. Such a labeling is D-constant: The two vertices of each diameter of the hexagon add to $D = b + g = d + h = a + e = 45 - K - L$. Moreover, $b = h + i + j$ and $g = a + c + j$, which imply that $D = b + g = h + i + 2j + c + a \geq 10$, being the sum of five distinct digits. When we add the six vertices of the hexagon and the remaining four interior vertices, we get $45 = 3D + c + f + i + j \geq 30 + c + f + i + j$, implying that $c + f + i + j \leq 15$. Moreover, $6 \leq c + f + i + j$, because the sum of four distinct digits between 0 and 9 is at least 6. To these inequalities we add $3D$ to get

$$6 + 3D \leq 3D + c + f + i + j = 45 \leq 15 + 3D,$$

whence we obtain $10 \leq D \leq 13$. Next, note that

$$c + f + i = 3K + 3L - 90 - j \quad \text{and} \quad K = a + c + j + i + h \geq 10.$$

For every pair $(D,\ j)$ with $10 \leq D \leq 13$ and $0 \leq j \leq 9$, we calculate in Table 3 the values of

$$K = D - j, \quad L = 45 - K - D = 45 - 2D + j$$

so long as $K \geq 10$. The cancelled cells (marked 'X') represent $K \leq 9$.

Table 3: Given $(D,\ j)$, a complete list of all viable values of $(K,\ L)$

| | $j$ | | | | |
|---|---|---|---|---|---|
| $D$ | 0 | 1 | 2 | 3 | 4 |
| 10 | (10, 25) | X | X | X | X |
| 11 | (11, 23) | (10, 24) | X | X | X |
| 12 | (12, 21) | (11, 22) | (10, 23) | X | X |
| 13 | (13, 19) | (12, 20) | (11, 21) | (10, 22) | X |

For each viable combination of $(D,\ j,\ K,\ L)$ listed in Table 3, in Table 4 we try to reconstruct the labeling until we reach either a contradiction or a success. To do so, first we construct every triplet that adds up to $K$; then we pick two non-overlapping such triplets, if possible. The central vertex is already assigned digit $j$; and the remaining three digits necessarily add up to $L$, the induced label of the third triangle. Next, we must pick one element from each triangle triplet, if possible, to form the labels $(c,\ f,\ i)$ of the interior vertices. Finally, we consider all $2^3$ combinations of the labels of the remaining vertices

Table 4: Reconstructing all labeling with face sums $(K,\ K,\ K,\ L,\ L,\ L)$

| $D$ | $j$ | $K$ | $L$ | Triangle Triplets | $c+f+i$ | $c, f, i$ | Solution |
|---|---|---|---|---|---|---|---|
| 10 | 0 | 10 | 25 | (712, 613, 514, 523) | 15 | x | |
| 11 | 0 | 11 | 23 | 713-524-689 | 12 | 1, 2, 9 | 731-542-869-0 |
| | | | | | | 1, 5, 6 | 731-245-896-0 |
| 11 | 1 | 10 | 24 | 604-523-789 | 11 | 0, 3, 8 | 640-253-798-1 |
| | | | | | | 0, 2, 9 | 640-352-789-1 |
| 12 | 0 | 12 | 21 | 912-534-678 | 9 | x | |
| | | | | 813-624-579 | 9 | x | |
| | | | | 723-615-489 | 9 | x | |
| 12 | 1 | 11 | 22 | 704-623-589 | 8 | 0, 3, 5 | x |
| | | | | 803-524-679 | 8 | 0, 2, 6 | 830-542-976-1 |
| 12 | 2 | 10 | 23 | 703-514-689 | 7 | 0, 1, 6 | 730-451-986-2 |
| 13 | 0 | 13 | 19 | 913-724-568 | 6 | 1, 2, 3 | x |
| | | | | 913-625-748 | 6 | x | |
| | | | | 823-715-946 | 6 | x | |
| | | | | 814-625-379 | 6 | 1, 2, 3 | 841-652-973-0 |
| | | | | 715-634-982 | 6 | 1, 3, 2 | 751-892-463-0 |
| 13 | 1 | 12 | 20 | 903-624-578 | 5 | 0, 2, 3 | x |
| | | | | 804-723-569 | 5 | x | |
| | | | | 705-624-389 | 5 | 0, 2, 3 | 750-462-893-1 |
| 13 | 2 | 11 | 21 | 803-614-579 | 4 | 0, 1, 3 | x |
| | | | | 713-605-489 | 4 | x | |
| 13 | 3 | 10 | 22 | 802-514-679 | 3 | 0, 1, 2 | x |
| | | | | 604-712-589 | 3 | x | |

of the triangles and verify if any combination is a solution to the puzzle. Below we give a little more detailed explanations of how Table 4 is constructed.

For example, in the top row $(D,\ j,\ K,\ L) =\ (10, 0, 10, 25)$, and the four triangle triplets that add up to 10 are 712, 613, 514, 523. But no two of these triplets are non-overlapping. Hence, we cannot form the three triangles. On the other hand, in the second row $(D,\ j,\ K,\ L) =\ (11, 0, 11, 23)$, and we can pick only one pair of non-overlapping triplets 713 and 524, each adding to 11; the remaining digits 6, 8, 9 add to $L = 23$, as desired. Next, we ensure $c+f+i = 12$ by picking one element from these triplets as $(1,\ 2,\ 9)$ or $(1,\ 5,\ 6)$. Each choice leads to a solution to Puzzle 2.2. We leave the explanation for the other rows to the astute reader.

Table 4 exhibits 9 solutions to Puzzle 2.2 satisfying $c < f < i$. We leave it to the reader to draw a diagram for these solutions. Each solution, after rotation and reflection, yields 6 solutions, for a total of 54 solutions to Puzzle 2.2.

#### 10.2.3 Two Triangles and a Non-intermediate Pentagon Receive Equal Induced Labels, So Do the Other Three Faces

Lemma 3. There is no solution to Puzzle 2.3.

Proof. The proof is by contradiction. Suppose that, if possible, a vertex labeling induces the same face label $M$ to two triangles ABC, GHI and one of the non-intermediate pentagons, say BCJFD, and the same label $N$ to the remaining three faces. Comparing the label $N$ of triangle DEF and pentagon EFJIG, and eliminating the labels of the common vertices E and F, we have $d = j + i + g$. Likewise, comparing the label $M$ of triangle ABC and pentagon BCJFD, and eliminating the labels of the common vertices B and C, we have $a = j + f + d$, whence $a = 2j + f + i + g$. Hence, the induced label of triangle ABC is

$$M = a + b + c = 2j + f + i + g + b + c \geq 15$$

since the six symbols are distinct between 0 and 9, and the induced label of pentagon ACJIH is

$$N = a + c + j + i + h = 3j + f + 2i + g + c + h \geq 16$$

because the six symbols are distinct between 0 and 9; and even if we associate the higher coefficients to multiply the lower-valued symbols, the sum is at least $3 \times 0 + 2 \times 1 + 2 + 3 + 4 + 5 = 16$. Combining the above two inequalities, we note that $2M + N \geq 46$.

On the other hand, the induced labels of the three triangles add to $2M+N = 45 - j \leq 45$. Thus, we reach a contradiction; and we must conclude that Puzzle 2.3 has no solution. □

## 10.3 D-Constant Labeling

In solving Puzzle 2 we have extensively used the D-constant property of a labeling. Let us solve the exercise posed in [8] regarding the number of D-constant labeling of a half-cube.

When each of the three diagonal sums equals $D$, the induced label of the hexagon is $3D$, which being the sum of six distinct digits is between $0 + 1 + 2 + 3 + 4 + 5 = 15$ and $4 + 5 + 6 + 7 + 8 + 9 = 39$. Hence, $5 \leq D \leq 13$. For each such $D$, we shall determine how many ways we can choose the three non-overlapping pairs of digits adding up to $D$. For instance, we can choose the three non-overlapping pairs of digits adding up to $D = 5,\ 6,\ 12,\ 13$ in exactly one way each; adding up to $D = 7,\ 8,\ 10,\ 11$ in exactly $\binom{4}{3} = 4$ ways each; and adding up to $D = 9$ in exactly $\binom{5}{3} = 10$ ways. Therefore, three non-overlapping pairs each adding up to $D$ can be chosen in $4 \times 1 + 4 \times 4 + 1 \times 10 = 30$ ways. Next, for each such choice, the pairs can be assigned to the three diameters of the hexagon in 3! ways, and then the end vertices of each diameter can be interchanged in two ways. Therefore, to achieve a D-constant property, the six vertices of the hexagon can be labeled in $3!2^3 = 48$ ways. Finally, the remaining four digits can be assigned to the remaining four vertices (interior to the hexagon) in $4! = 24$ ways. Therefore, we have $30 \times 48 \times 24 = 34560$ D-constant labeling.

## 10.4 Summary and New Explorations

We considered a hexagonal half-cube obtained by letting a plane pass through the center of a cube and orthogonal to a diagonal. A hexagonal half-cube has three triangles, three pentagons and a hexagon as its faces. When the ten vertices of a half-cube are labeled with numbers, then each face is said to receive an induced label given by the sum of the labels of all its vertices. The objective has been to label the vertices using digits 0 through 9 exactly once so that the six faces (other than the hexagon) receive induced labels that are "two distinct numbers with frequencies three each". We described how to find all such 12+54+0=66 labeling using mathematical logic different from a complete search of all 10! permutations. Also, we simplified the discovery of all 12 solutions to Puzzle 1 studied in [8].

We hope mathematics teachers in high schools and colleges will modify Puzzle 2 to inspire young students to explore mathematics on their own. Simply by changing the objective of the vertex labeling (the last seven words in the statement of Puzzle 2), one can pose a large variety of puzzles. For instance, students may be asked to list all labeling of the ten vertices of the hexagonal half-cube that induce (1) the same label to five faces while the sixth face label is different, (2) the same label to four faces while the remaining two face labels are also equal, (3) three pairs of face labels, and (4) six face labels that are (i) consecutive integers, (ii) consecutive even numbers, or (iii) consecutive prime numbers.

Need we mention more variations to this puzzle? Let us empower you to make your own variations: Randomly assign the ten digits to the ten vertices—but keep no record of the assignment. Then compute the six induced face labels; sort them and record the sorted face labels—keeping no record of which face labels were associated with the triangles and which with the pentagons. Now there is no turning back, erase the vertex labels for good—both from paper and from memory. A few days later, label the ten vertices to recover the six recorded (sorted) induced face labels, or ask your colleagues/students to do so. You know that there is a solution. Can you, your colleagues and your students recover not only that one solution (of course, you will not recognize it since you have no record of it), but also all solutions?

We offer the above variations as examples of open-ended problems in response to Ramanujam's question in [6]: "Can we have problems that start with hands-on activities and constructions (perhaps based on trial and error) that lead up the ladder of abstraction into esoteric conjectures and proofs?"

Finally, we urge astute readers to pose and solve other puzzles on the hexagonal half-cube, or any other nice graph such as the ones studied in [9] and [2]. It is a good idea to consult [3] to check if the discovery is new or already documented. Also, readers and their students may try to find alternative proofs to the results established in this paper. We await listening to stories narrated by mathematics teachers of their students' engagement, struggles, challenges, and occasional triumphs as they 'play' with such puzzles.

To conclude this paper, once again we quote from Jayasree [5]:

> "Provided an opportunity, students do engage in these processes enthusiastically and come up with questions of their own to explore. Whether all answers to the questions posed are arrived at or not, it is important that the questions be raised, and options explored. This gives students a taste of what it means to 'do' mathematics."

We agree whole-heartedly; we are committed to providing such opportunities; we trust you too will do your parts in this noble outreach.

## Acknowledgment

I thank my student (turned collaborator) Colin Tully for inquiring if for the hexagonal half-cube there are labeling puzzles other than TP-constant labeling. This work resulted from satisfying his curiosity. Professor George Gopen, who, through his books and workshops, taught me how to write in a reader-centric style, deserves my lifelong gratitude. The purpose of this paper was reformulated based on the Editor's timely tips and guidance, which I acknowledge gratefully.

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# 11 Mathematics – Art that would rather be science by M. S. Raghunathan

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Mathematics seems to have acquired an identity as an independent intellectual discipline early on in human history. More: even in ancient times it was valued very highly as this Sanskrit couplet dating back to the seventh century BCE indicates:

यथा शिखा मयूराणां नागानां मणयो यथा।
तद्वद् वेदांगशास्त्राणां गणितं मूर्धनि स्थितम्॥

"As are the crests of the peacocks, as are the jewels of the serpents,
So is mathematics at the helm of all Sciences."

The Greek philosopher Plato had this to say of Mathematics: "The highest form of pure thought is in mathematics." The same sentiments were voiced by Carl Friedrich Gauss, one of the greatest mathematicians of all time, two and a half millennia later: "Mathematics is the Queen of Sciences". Gauss added a rider to that to which we will be return later.

Figure 1: Carl Friedrich Gauss 1777-1855

Mathematics displays indeed many qualities of the royal personage – aloofness for one. It is certainly considered a science, yet it stands apart from the other sciences: the very title of Newton's magnum opus – Philo-sophiae Naturalis Principia Mathematica (Mathematical Principles of Natural Philosophy) – suggests this aloofness. It is mathematical intervention that decisively confers the label "science" on an intellectual discipline, conferment of an honour as it were, by a queen.

The queen in history, with rare exceptions like Elizabeth I of England or Catherine the Great of Russia, has wielded very little power. She has been a mere decoration, which perhaps explains why that claim for mathematics by Gauss has seldom been challenged by non-mathematician scientists! Mathematics can perhaps be said to be self-indulgent – the problems that (most) mathematicians choose to work on have little to do with the practical world; and that choice is the result of the mathematician's fancy.


Figure 2: The queen in the nursery rhyme

Even in the nursery rhyme, the queen is engaged in the frivolous act of eating bread and honey, while the king attends to the serious business of "counting out his money"!

Queens are supposed to be whimsical and so is mathematics. Its development is the result of the whims and fancies of its practitioners; and there it resembles art. Yet it is not quite as arbitrary as the queen in Alice in Wonderland. Incidentally, Lewis Carol was the pen-name of Rev. Ralph Dodgson, a mathematician, and a lot of the humour in the book has its origins in (mathematical) logic.


Figure 3: The Queen of Hearts in Alice's Adventures in Wonderland.

But the quality that popular imagination ascribes most to the queen is beauty. The queen in Alice in Wonderland in that illustration is of course ugly. "Good Queen Bess" was not known for her looks; her rival whom she ground to dust – Mary, Queen of Scots – was on the other hand a handsome woman (and that perhaps contributed to Elizabeth's dislike of her cousin). However, queens in mythological stories are more or less always beautiful. Sita in the Ramayana was the most beautiful woman in the world. In the Mahabharata, Pandu has two queens, one of whom, Madri, was a renowned beauty. In Homer's Illiad again, Helen, the queen of Sparta is the most beautiful woman in the world. And Mathematics has that quality of the queen too: it is beautiful.

Figure 4: Helen, Queen of Sparta

But what is beauty? It is indeed difficult to define. The opening line of Keats famous poem "Endymion" perhaps provides the best definition of beauty: "A thing of beauty is a joy forever". The mathematician will readily agree with that; but that endorsement would be inspired by another product of Greek genius – Euclid's "Elements" – not the mythological story of Endymion.

Figure 5: Endymion and Selene.

Euclid's "Elements", has delighted generations since the third century BCE. The early theorems in Elements do have some relevance to the practical world, but the later ones seem to be the results of an aesthetic drive – a search for the beautiful. And in that again mathematics resembles art.

Here is what a leading public intellectual of the 20th century, Bertrand Russell, has to say about mathematics. "Mathematics rightly viewed possesses not only truth but supreme beauty – beauty, cold and austere like that of sculpture without appeal to our weaker nature, without the gorgeous trappings of paintings or music and capable of stern perfection such as only the greatest art can show."

Russell was a mathematician though he was much better known as a philosopher, writer and peace activist. He was awarded the Nobel Prize for Literature in 1950.

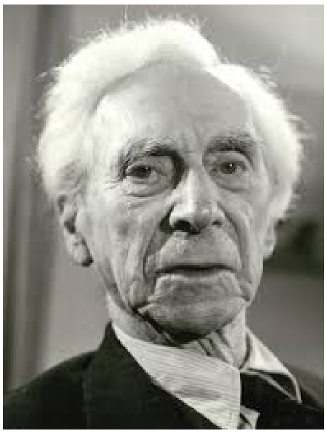

Figure 6: Bertrand Russell 1872–1970.

But the beauty possessed by mathematics is recognized by men in other scientific disciplines as well. It is generally believed that biologists are far from being fond of mathematics, but here is one who has no doubts about mathematics possessing beauty: "For the harmony of the world is made manifest in Form and Number and the heart and soul and poetry of Natural Philosophy are embodied in the concept of mathematical beauty."

Figure 7: D'Arcy Wentworth Thompson 1860-1948.

Yes, Mathematics is beautiful. In fact a lot of all science is motivated by the aesthetic quest, but mathematics, much more than other sciences. Here is what the great Newton says: "I do not know what I may appear to the world; but to myself I seem to have been one like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, while the great ocean of truth lay all undiscovered before me."

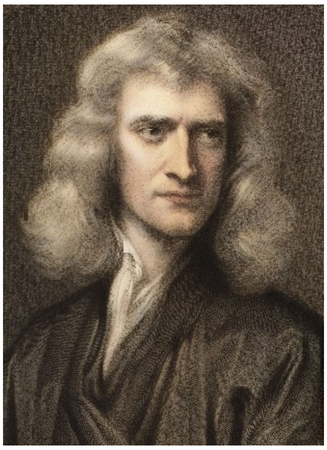

Figure 8: Isaac Newton 1643-1727.

Well that was a bit disingenuous – Newton surely knew that all of Europe recognized him as the greatest intellect of his times; so the first sentence shouldn't be taken seriously. It is (unfelt) regulation modesty; but his modesty before the "ocean of truth" was genuine. Newton however was as much a mathematician as a physicist and in fact to apprehend the beauty of his physics, mathematics is a necessary tool.

Not only in Newton's work, but beauty in much else in science too needs the help of mathematics for its appreciation. For, as Galileo put it, mathematics is the language in which the "book of nature" is written.

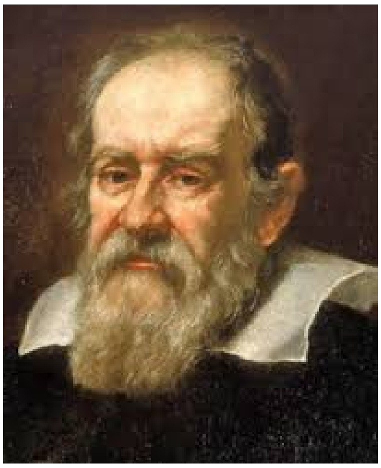

Figure 9: Galileo Galilei 1564-1632.

It is generally agreed that all art is pursuit of beauty; and under that title mathematics surely qualifies as art. But beauty, it is said, is in the eye of the beholder. Here is a beholder – the man whose words I suggested can be used to define beauty – to whom science is ugly. In his poem "Lamia" he tears into Newton in particular:

> ...Do not all charms fly
> At the mere touch of cold philosophy?
> There was an awful rainbow once in heaven:
> We know her woof, her texture; she is given
> In the dull catalogue of common things.
> Philosophy will clip an Angel's wings,
> Conquer all mysteries by rule and line
> Empty the haunted air, and gnomed mine–
> Unweave a rainbow, ...

Newton, for Keats, is a philistine vandal who tore apart the rainbow!

Figure 10: John Keats 1795–1821.

But there is another poet – Alexander Pope who wrote "God said 'Let Newton be'. And all was light!" You cannot pay a more handsome tribute to Newton than that take-off on the Bible.

All mathematics began with counting and there is no gainsaying its importance in our lives. One may say (with apologies to Descartes) "I count, therefore I count". And there is beauty in counting, a beauty we normally fail to see, thanks to the "anaesthetic of familiarity" to borrow an expression from Richard Dawkins; but its presence is attested by the pleasure some children take in counting.

There can be little doubt that counting was born in the primitive market place where one had to set relative values of commodities (like ten pumpkins to one goat etc.). But what underlies counting is profoundly abstract.

Abstraction consists on the one hand of recognizing something common in diverse phenomena and, on the other hand, ignoring the irrelevant in a given investigation. One sees for example that the mathematics of compound interest is essentially the same as that of radioactive decay – an illustration of the first aspect of abstraction. Ignoring friction in the first instance in the study of motion is the first step in turning mechanics into a (mathematical) science.

Counting seems almost instinctive but what underlies it is profound abstraction displaying both aspects of abstraction. The human mind recognizes that there is something common to a whole lot disparate collections – the number of members of the collection: for example the number 5 is what is common to the collection of fingers in a man's right hand, the number of toes in a woman's left foot, the Pandava brothers, the "elements" (earth, water, sky, wind and fire) that make up our universe according to the philosophers of yore and so on. The second aspect comes through in that we completely ignore the individual characteristics of the members of the collection – it does not bother us that the big toe is so different from the small toe, that Bhima is lot bigger than the rest of the Pandavas and so on.

Ovid, 2000 years ago, said "ars est celare artem" – art lies in concealing artificiality. But that dictum has been discarded long since and modern art embraces abstraction (which modifies and distorts subjects it represents). So abstraction does not distance mathematics from art.

As I said, Euclid's elements qualifies as beautiful because it is a "joy forever" There is some mathematics – in number theory – due to Euclid that is considered by most – if not all – mathematicians as a lovely piece of mathematics and it is elementary enough to be accessible to anyone with no more than a background of school mathematics. I will now describe the result of Euclid and its proof (due to him).

If you find the result and its proof beautiful, your aesthetic sensibilities will find the right resonance in the mathematical mind.

Figure 11: Euclid circa 300 BCE.

The question that Euclid asked himself was whether the collection of all prime numbers is finite and he answered it in the negative. Recall that a number is a prime if its only divisors are 1 and itself. His argument runs as follows. Suppose that there are only finitely many primes. They can then be listed as $p_1, p_2, \ldots, p_n$. Consider now the number $N = p_1 p_2 \cdots p_n + 1$. This number cannot be a prime as it is greater than all the $p_i$. So it has a divisor $d$ with $1 < d < N$. Let $d_0$ be the least among its divisors other than 1 and itself. As any divisor of $d_0$ is also a divisor of $N$, $d_0$ is a prime. So $d_0 = p_i$ for some $i$ between 1 and $n$. Now dividing $N$ by $p_i$ leaves a remainder of 1; so $p_i = d_0$ is not a divisor of $N$, a contradiction. Thus our assumption that the number of primes is finite is not true. Hence the number of primes is infinite.

All of you are no doubt aware that the concept of zero as a number originated in this country as also its use for writing down numbers in the place value scheme. The Bakshali manuscript is the oldest mathematical text containing the symbol for zero and is dated to around the 4th century CE by most historians of mathematics. It has also numbers written in the place value system. The

Bakhshali manuscript written on birch bark was found in 1881 in the village of Bakhshali near Peshawar in Pakistan.

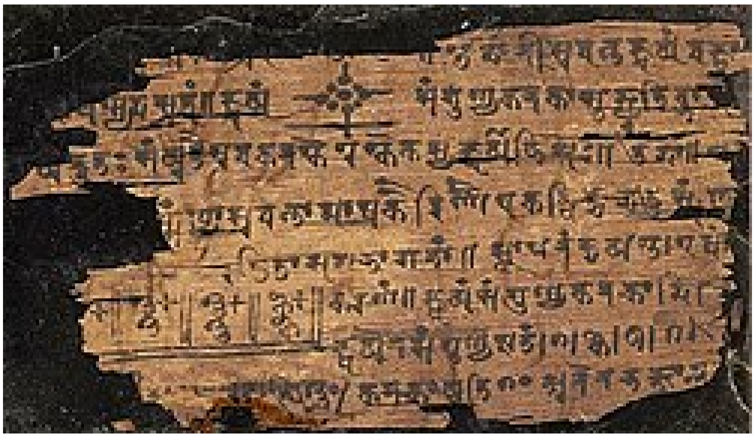

Figure 12: A fragment of the Bakshali manuscript.

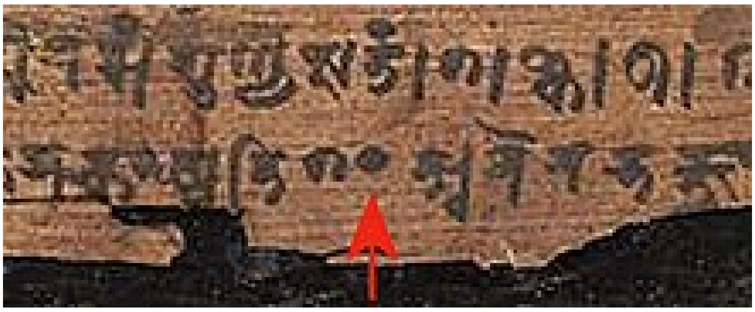

Figure 13: The numeral ‘zero’ is seen here.

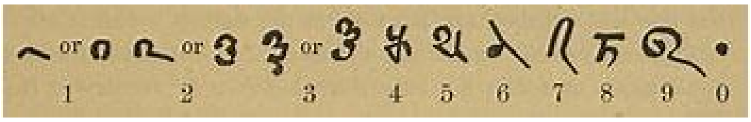


Figure 14: All the numerals from zero to nine.

It is natural to believe that the invention of the place-value system was the result of necessity, but I would suggest that it is the aesthetic urge not necessity, that led ancient India to this discovery. The place value system is really needed only when one has to deal with writing down and manipulating big numbers; and that need would have been felt by the Greeks and Persians who had organized huge armies some five hundred years and more before the Indian invention of the place-value system: and they were content enough with the clumsy representations available to them. Our ancestors on the other hand were obsessed with large numbers – vedic literature has names for powers of ten up to the nineteenth! Also they could name numbers orally following the place value system (which did not need the zero). There would seem to have been an urge in our mathematicians to find a nice method for writing down any number and I would consider that a quest for the beautiful.

One readily admits that all art is search for the beautiful; a lot of (beautiful) mathematics too is the result of the quest for the beautiful; and in that mathematics is akin to art. The natural sciences get their inspiration from natural phenomenon – they want to understand the how and why of nature's unfolding. Mathematics on the other hand is driven more or less by an internal dynamic – a contemplation of mathematical structures themselves. And it would appear that great mathematicians value the mathematics born of the aesthetic drive more than that which lends itself to the service of the external world. Admonished by the great mathematician Fourier for pursuing useless mathematics, his greater contemporary Jacobi responded with a spirited defence of aesthetics-driven mathematics: "A scientist of Fourier's calibre should know the true end of science is the greater glory of the human mind, and under that title, a question about numbers is worth as much as a question about the system of the world."

Figure 15: Jean-Baptiste Joseph Fourier 1768-1830 and Carl Gustav Jacobi 1804-1851.

Mathematicians are all the time making efforts to understand mathematical constructs themselves in greater depth looking for hidden beauties in them. This is rather like a sculptor shaping a piece of amorphous stone into a beautiful figure and there is an element of subjectivity in both. The artist chooses the stone and turns it into a shape that he/she fancies while the mathematician chooses the mathematical structure he/she wants to examine and decides what to do with it. Here are two examples of sculpture that match "Elements" in drawing adulation after two millennia and more.

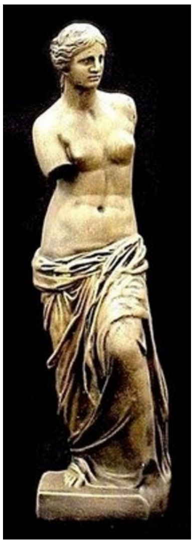

Figure 16: Venus of Milos and the Ashokan Lion Capital.

We will presently see more examples of aesthetics as the driving force behind mathematics. But not all mathematics is the result of this kind of internal dynamic. Newton developed the calculus as a tool to understand motion; and as I had pointed out earlier much of the arithmetic we learn at school was born in the market place. Nevertheless, even mathematics born outside its fold often takes on a life of its own and develops through an inward exploration.

I will now embark on some historical episodes where internal dynamics driven by aesthetics was at work effecting progress in mathematics. Linear equations seem to have already appeared on the mathematical scene by 1500 BCE as one can divine from the clay tablets excavated in Iraq. And this may well be the result of practical considerations. But one cannot see what practical reasons could have motivated these ancient people into thinking about the quadratic equations – the clay tablets have something on them as well. In any case the quadratic equation was pursued elsewhere too and by Aryabhata's time (6th century CE) we had arrived at the present solution of the quadratic equation (which we learn at school). While solving quadratic equations turned out to be useful in the natural sciences, higher degree equations have had no such impact; nevertheless the mathematician wanted to find solutions of the cubic equation. The search was on all across the world and the motivation was aesthetics.

There were unsuccessful efforts made by mathematicians in India as well as the middle East and China to solve the cubic. Omar Khayyam who is famous as a Persian poet and who was in fact also a leading mathematician of his times was one of them; Bhaskara II of India (the author of Leelavati) and Sharaf al-Tusi, a Persian also tried their hands at it. Bhaskara was not a full-fledged poet but in his Leelavati all problems posed are in the form of poems. And he shared his year of birth as well as death with al-Tusi!

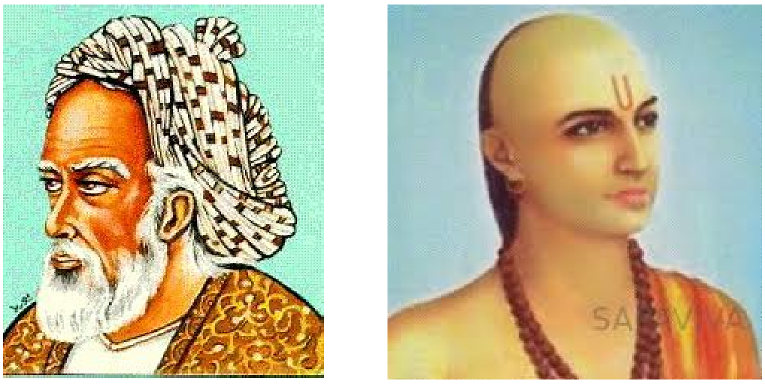

Figure 17: Omar Khayyam 1048-1131 and Bhaskara II 1114-1185.

It took another 300 years before the solution of the cubic equation was achieved. The man who put the seal on the problem was Niccolo Tartaglia while another Italian, Scipione Del Ferro (1465 - 1526) had earlier given the solution under some restrictions on the cubic equation. An exposition of both Del Ferro's and Tartaglia's works was made by Gerolamo Cardano (1501 - 1576), an important renaissance intellectual. Soon after Tartaglia's work became known, Lodovico Ferrari, a disciple of Cardano building on the work of Tartaglia came up with the solution of a quartic (i.e., 4th degree) equation. But taming the cubic and quartic only led to mathematicians asking "What about the quintic" - the fifth degree equation; and sure enough there was frenetic activity to find a solution of the quintic equation.

Figure 18: Niccolo Fontana Tartaglia 1499-1557 and Lodovico Ferrari 1522-1565.

I must first clarify what is meant by finding a solution of a polynomial equation. What mathematicians were looking for is an expression of the root in terms of the coefficients together with all kinds of numbers that come out of them using the standard arithmetical operations and extracting square roots, cube roots, fourth roots etc., the process being repeated indefinitely. In technical parlance one calls it finding a solution by radicals. For some two centuries this activity went on till Henrik Abel, a young Norwegian mathematician came up with a shocker. He showed that there are fifth degree equations none of whose roots lie in the collection of numbers generated as above from the coefficients leave alone being able to write down an expression for the root in terms of these numbers! And that naturally gave a totally unanticipated direction to the development of Mathematics.

Another young man, French this time, Evariste Galois, stepped in and developed a beautiful theory that could decide whether any given equation of any degree whatever can be solved by radicals. Both Abel and Galois wanted to understand mathematical structures better and did not have any application to the practical world in mind.

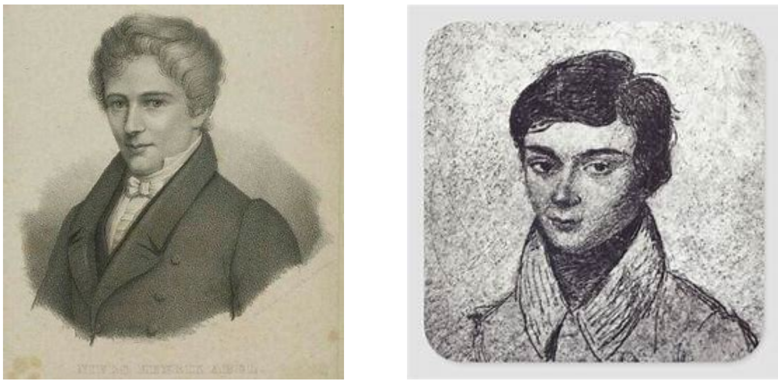

Figure 19: Niels Hendrik Abel 1802-1829 and Evariste Galois 1811-1832.

Algebraic equations belong in algebra, the manipulation of variable quantities. When one restricts one's attention to the variable quantity staying within the world of integers it acquires another name – (higher) arithmetic or number theory. Gauss, when he said that mathematics is the queen of sciences also added that "Number Theory is the queen of mathematics'. Evidently Euclid's theorem on the infinitude of primes belongs in Number Theory. Modern number theory has its origins in Mesopotamian mathematics. A triple of natural numbers (or whole numbers) $(a, b, c)$ is a Pythagorean triple if $c^2 = a^2 + b^2$. The name Pythagorean triple derives from the fact that $(a, b, c)$ would be the 3 sides of a right angled triangle as guaranteed by Pythagoras's theorem. It is well known – and easy to see – that (3,4,5) and (5,12,13) are Pythagorean triples. The Babylonians knew of many more as the tablet below excavated in Iraq shows.

Figure 20: Mesopotamian list of Pythagorean triples.

The Greek number-theorist Diophantus asked if one could determine all Pythagorean triples and in fact listed them all: if $p$ and $q$ are natural numbers with $p > q$ then $(2pq, p^2 - q^2, p^2 + q^2)$ is a Pythagorean triple; moreover every Pythagorean triple is either of this form or of the form $(pq, (p^2 - q^2)/2, (p^2 + q^2)/2)$ with both $p$ and $q$ odd. And this result is to be found in his book "Arithmetica". That the triples described are Pythagorean is easy to prove – it is an exercise in High School algebra.

Pierre de Fermat was a judge in a provincial court in the town of Toulouse in 17th century France and apparently a competent one at that. Mathematics was however an obsessive and passionate hobby to which he devoted all his spare time and he ended up as one of the all-time greats in mathematics. He owned a copy of Diophantus' Arithmetica and it was his habit to record his own thoughts in short notes on the margins of the book. One such note was to make mathematical history. Fermat was secretive and the mathematical community came to know of this note on the margin only after Pierre's son Samuel published an annotated version of Arithmetica with Fermat's marginal notes. Every one of those notes, most of which formulated results without proofs, fascinated mathematicians and people set out to find proofs. Within a couple of decades all the unproved assertions in the marginal notes had been proved except the one below appearing in the annotated Arithmetica.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

Rursus oporteat quadratum 16 diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. $\frac{16}{5}$ erit

Figure 21: Pierre de Fermat 1607-1665 and his marginal note in the annotated Arithmetica.

That note in Latin translates as: On the contrary, it is impossible to separate a cube into (the sum of) two cubes, a fourth power into two fourth powers, or, generally any power above the second power into powers of the same degree. I have discovered a truly marvellous demonstration (of this general theorem) which this margin is too narrow to contain.

The hunt for a proof of the theorem started and generations of outstanding mathematicians went after it. But all it produced in two centuries was incremental progress. Digging into Fermat's papers, Euler found that Fermat essentially had a proof in the case $k = 4$ and then went on to give a proof for the case $k = 3$. That was all the progress made in the 18th century. And in the 19th century only two more numbers, 5 and 7, yielded.

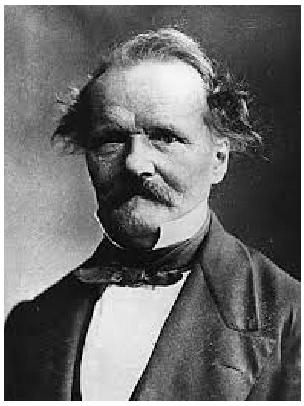

Figure 22: Leonhard Euler 1707-1783, Sophie Germain 1776-1831 and Ernst Kummer 1821-1893.

Sophie Germain had some interesting ideas which held out a promise of proving the theorem for a whole lot of integers in one fell swoop, but as it turned out the ideas eventually helped only in the case $k = 5$ (Dirichlet used her ideas). The mathematician who made the most impressive progress was Ernst Kummer. He developed a general theory and was able to prove the theorem for all numbers up to 100. Further, the beautiful general theory he developed helped solve several other problems in number theory even while throwing up a host of new interesting problems.

It is towards the end of the last century (in 1994) that Fermat's last theorem was finally proved by a British mathematician Andrew Wiles working in Princeton University. But some ten years before that a German mathematician Gerd Faltings showed that for any $k \geq 3$ there are only finitely many triples $(a, b, c)$ of natural numbers admitting no common factor other than 1 such that $a^k + b^k = c^k$, which is evidently great progress towards the theorem. However Andrew Wiles made no direct use of Falting's work. His proof of Fermat's Last Theorem was a tour de force making use of a lot of mathematics developed during the 20th century (some of which is due to Faltings). One cannot help the conclusion that Fermat was mistaken in thinking that he had a proof of the theorem. A lot of mathematical development thus had its origin in the fancy of one mathematician – Fermat. But of course that fancy had to have a resonance in the mathematical mind in general.

I mentioned Galois theory. That theory and its ramifications that followed also were very much involved in the proof of Fermat's last theorem. Galois theory is in some sense the very foundation of Number Theory. Mathematical structures called groups play an all important role in Galois theory; and Galois himself contributed a great deal towards the development of the theory of groups. But the theory of groups has come to play a big role in differential geometry. In fact, Felix Klein, one of the leading figures during the decades at the turn of the last century pointed out that the study of geometry is in essence the study of invariants under various kinds of groups. Group theory, though developed for purely mathematical reasons, has found lots of applications in Physics and Chemistry.

A contemporary of Fermat, René Descartes had an equally profound influence on the direction mathematics took in the mid 17th century. Till Descartes, geometry and algebra followed distinctly different paths, the very nature of the arguments used seemingly very different. Descartes invented what we call analytic geometry and pressed algebra into the service of geometry; conversely geometry could contribute to solving problems in algebra.

Figure 23: René Descartes 1596-1650.

And in fact this had a big role in the proof of Fermat's Last Theorem. Descartes' motivation was simply to widen the study of geometry to include shapes that could not be handled by the prevalent methods which went back to the Greeks. That points on the plane can be represented by a pair of real numbers, the coordinates, and so curves in the plane can often be thought of as a collection of points whose co-ordinates are related by an algebraic equation; and the study of geometry of the curve becomes the study of the algebraic equation. Descartes' motivation was the renovation of geometry. But his great idea has been an enabler for diverse human endeavours by geometric representation of a pairs of quantities related to each other: graphs are an indispensable tool in all kinds of human activities. Needless to add, Descartes had no conception of the existence of this character in the cartoon, leave alone wanting to help him! Nor did Galois have any idea that the Group Theory he contributed would prove useful outside mathematics.

Figure 24: Eugene Wigner 1902-1955

This phenomenon where a mathematical theory founded for purely mathematical reasons finds applications in diverse human activities is discussed by the Nobel Laureate Eugene Wigner in a lecture (which appeared later as an article in a mathematics journal) titled "The unreasonable effectiveness of mathematics in the natural sciences".


Figure 25

Another example of such "unreasonable effectiveness" is the use of Riemannian Geometry in Einstein's Relativity. Bernhard Riemann developed what has come to be called Riemannian Geometry mainly to expand the scope of Geometry to include many more kinds of geometric shapes beyond what Descartes could do – it was essentially an aesthetic quest; his interest in physics was peripheral and he could not have anticipated the role it was to play some 50 years after his discovery.

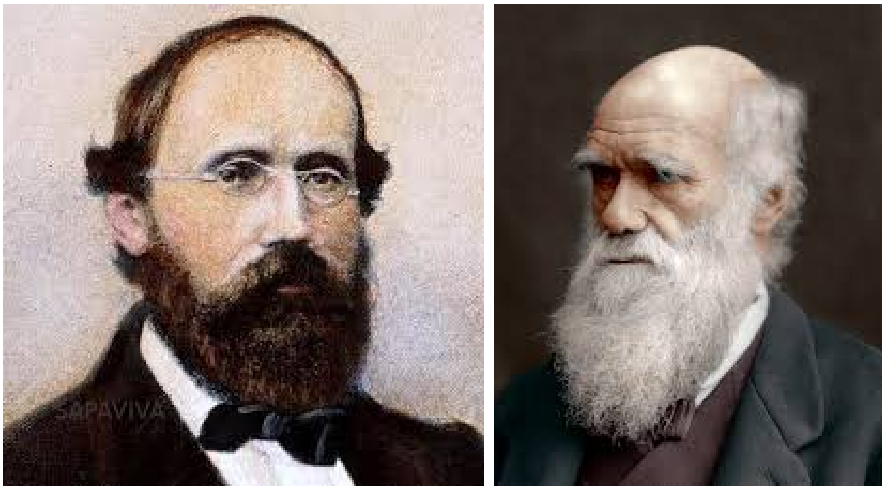

Figure 26: Bernhard Riemann 1826-1846 and Charles Darwin 1809-1872.

Charles Darwin takes a stand almost diametrically opposite that of Wigner': "Every new body of discovery is mathematical in form because there is no other guidance we can have". Personally I am inclined towards Darwin's views on the subject. Natural sciences seek to understand natural phenomena and they have the unstated belief that there has to be a beautiful explanation, an explanation that can eventually be made intelligible to lay people. There is I think a close connection between the beautiful and the intelligible. An idea to be beautiful needs first to be intelligible to someone. Viewed that way, it should not be surprising that the pursuit of beauty which is very much the occupation of the mathematician should help make nature intelligible.

For the natural scientist, there can be a conflict between truth and beauty. Galileo developed a beautiful theory to explain the behaviour of tides, but it had to be discarded as its predictions were not in tune with the observed behaviour of tides. A more recent episode is the effort by Hermann Weyl a leading mathematician-physicist of the 20th century to understand gravitation. He developed a mathematical theory called Gauge theory for this purpose, but the theory failed to deliver. Here is the comment Weyl made in that context: My work has always tried to unite the true with the beautiful and when I had to choose one or the other, I usually chose the beautiful.

Figure 27: Herman Weyl 1885-1955.

May be that quote only suggests that Weyl was more mathematician than physicist. However the beautiful theory turned out to be helpful in understanding quantum chromodynamics! Gauge theory was mathematical truth and if it failed to help apprehend the truth about gravitation, it was able to intervene elsewhere in Physics. The mathematician is spared the dilemma of choosing between truth and beauty. The mathematician makes some basic assumptions called axioms (self-evident facts) and deduces from these through logical reasoning other statements, labelling them as “true statements”: but they are true only on the basis of the assumed axioms and are not “absolute truths”. If there is some inconsistency about the axioms, the entire edifice breaks down and there are no longer any true statements.

Scientists would readily agree that the pursuit of truth is a good job-description for their profession – in any case the greatest of them thought so. And it was for him at the same time – mark the words “smooth pebble” or “pretty shell” – pursuit of beauty. Curiously enough, Keats the science baiter ends another poem of his with the oft quoted lines “Truth is beauty and beauty truth…”, hoisting himself on his own petard! If one admits that truth is beauty where is the surprise in mathematics intervening in the natural sciences.

In every creative endeavour, there is a certain tension between imagination and discipline. In most of them the discipline is imposed from the outside. In the natural sciences, imagination is tempered by having to subordinate itself to the actual observations of nature. No matter how imaginative a theory, it has to be discarded if its conclusions run counter to the observations of the phenomenon it seeks to unravel. In mathematics the discipline is largely internal: mathematics has to meet the demands of rigorous deductive reasoning (science too has its demands of reasoning, but of a much less rigorous kind and its role is secondary to compliance with observation). A second disciplining agent (which is not recognized as such by most people) is aesthetics. The mathematician's choice of problems to work on is dictated by aesthetics; and elegance in the proofs is much sought after even if it may not always be achieved. Also, there is a strong urge to arrive at a more elegant proof if the first one is ugly. Let me illustrate this with what happened in the world of astronomy. It is a popular perception that Ptolemy's earth-centric model of the planetary system is not valid. That is not true. One can make a perfectly valid mathematical model of the solar system with any object in it – in particular the earth – as fixed. The helio-centric model proposed by Copernicus has come to stay because of the elegance of its mathematics while that of the Ptolemaic model is complicated and clumsy. Aesthetics wins hands-down.

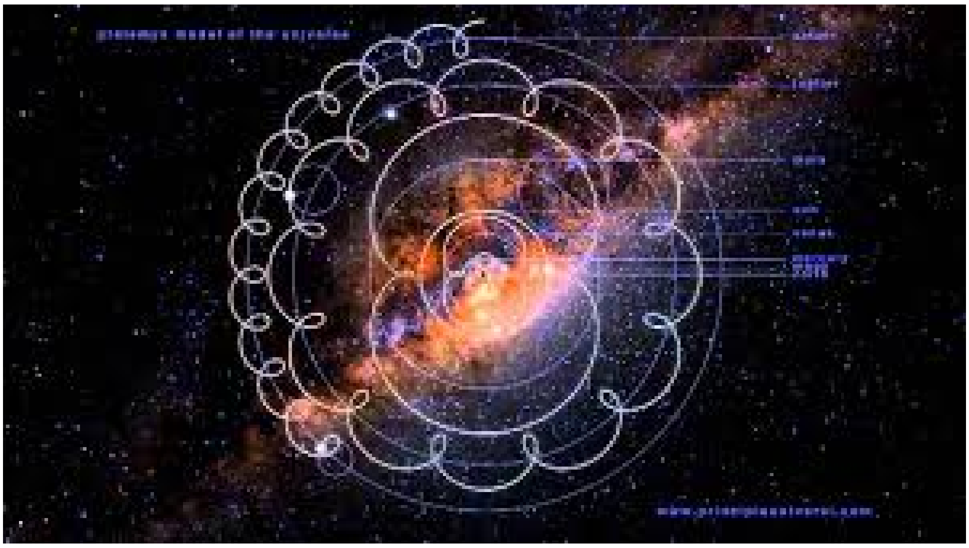

Figure 28: The planetary system in the Ptolemaic model.

When a problem in mathematics, perceived as difficult, gets solved, there is great excitement in the community of mathematicians. And of course the person who solved it is greatly admired; and that admiration is much greater if the solution is elegant. Elegance in this context is practically synonymous with being intelligible with ease. And the mathematician ever strives to produce elegant solutions to problems. After all, it is the gymnast one likes to watch, not the contortionist whose acts can be impossibly difficult.

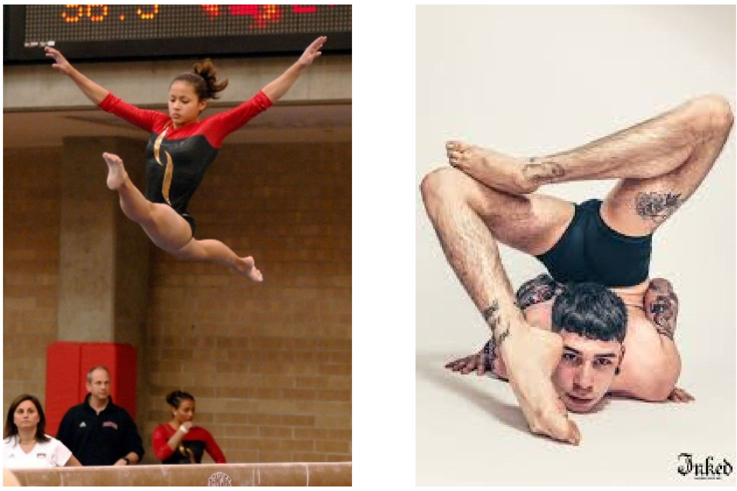


Figure 29: The gymnast and the contortionist.

Despite all these different ways in which mathematics is like art, making the case for it to be considered an art, mathematicians are more comfortable in the company of scientists rather than artists. There is one fundamental aspect where art and science differ and in that mathematics is on the side of science. Art often demands the "willing suspension of disbelief" while science asks for a deliberate suspension of belief. Also, as Galileo put it, mathematics is the language in which the book of nature is written. As they need mathematics in their work, scientists have a natural sympathy for the mathematician's pursuit. The artists on the other hand tend to regard mathematics as a formidable discipline which has little contact with the everyday world of their experience, except for the market place, which they view with derision. Attitude such as that of Keats is not uncommon and that obviously would put off the mathematician.

# 12 History in the Classroom: Area and Algebra by Amber Habib

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Abstract

Algebra and geometry generally stand apart in school mathematics. Prior to the introduction of coordinates in geometry, the principal interactions are the use of algebraic notation to express geometric formulas and relations, and of the quadratic formula to solve some problems related to length and area. In this note, we shall take up some well-known episodes from ancient India and Mesopotamia and suggest their use at certain stages of school to promote geometric manipulations as precursors to algebraic ones, hopefully leading to a more unified view of these topics.

## 12.1 Introduction

Geometry has the advantage over other parts of mathematics that its truths can become evident without first having to be translated into symbols. When we worry that students carry out formal manipulations without insight, our solution often is to suggest that geometric examples be used to provide intuition and motivation. Yet the manner in which this is done can relegate Geometry to a provider of examples on which symbolic manipulations show their power. Especially in the lower classes in school, Geometry involves cataloguing and classification rather than reasoning. For example, the Geometry syllabus for Classes I to V on the NCERT website is dominated by words like 'collect', 'sort', 'classify', 'observe', 'describe', 'identify' and 'distinguish'. This is in contrast to the 2006 position paper of the National Focus Group for the Teaching of Mathematics [9], where we find the following recommendations for the Primary stages:

> It is important also to build up a vocabulary of relational words which extend the child's understanding of space. The identification of patterns is central to mathematics. Starting with simple patterns of repeating shapes, the child can move on to more complex patterns involving shapes as well as numbers. This lays the base for a mode

> of thinking that can be called algebraic. A primary curriculum that is rich in such activities can arguably make the transition to algebra easier in the middle grades.

Area manipulations are visually convincing and can help accustom a child to mathematical reasoning from an earlier stage. We can use as a guiding principle the fact that two polygons have the same area if and only if they can be cut and reassembled into each other. This process of cutting and reassembling also happens to be, historically, the source of various basic processes of algebra. We shall give examples of this, beginning with purely geometric tasks and processes and then giving the corresponding algebra as well as suggesting appropriate stages of the curriculum where these can be discussed.

It will help if we keep in mind the current curriculum. Here are some relevant highlights for Classes III-VII [8]:

| | Geometry | Arithmetic & Algebra |
|---|---|---|
| Class III | Create shapes through paper folding and cutting. Count vertices, sides and diagonals. Explore tilings. | 3 digit numbers, place value and arithmetic operations, comparisons. |
| Class IV | Circles, tangrams, tilings, area and perimeter. | Multiple approaches to arithmetic operations, estimation, fractions: halves, thirds and fourths. |
| Class V | Tangrams, angles, create angles by paper folding, rotations and reflections. | Larger numbers, place value and arithmetic operations, multiples and factors, fractions: equivalence, decimal representation. |
| Class VI | Measure of angles, parallel and perpendicular lines, areas of squares and rectangles (whole number sides). | Negative numbers, number line, algebra: variables, unknowns, unitary method. |
| Class VII | Pythagoras theorem, congruence, areas of triangles and other shapes. | Rational numbers, powers, linear equations |

Our proposal is that we begin by discussing areas being 'same' or 'bigger/smaller' prior to the discussion of numerical values. For polygons, we can have the following definitions:

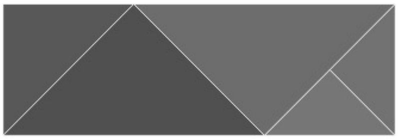
(a) Two polygons have the same area if one of them can be cut into smaller polygons which can be reassembled to be identical to the other.

(b) Polygon $A$ has smaller area than polygon $B$ if $A$ can be cut and reassembled into a shape which lies completely inside $B$.

We note that counting can be approached similarly. We can describe the concepts of 'same' and 'more/less' before developing numbers, just by pairing off elements of two collections. There is evidence that children can count in this sense well before they are taught numbers of any significant size. (See [11, p.3])

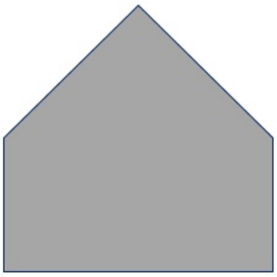
One can prepare the ground for this concept of area in Classes IV and V, using the tangram activities. The current activities require the student to create a given shape from the tangram pieces. All that is required is to add an initial portion where a certain shape is already made from tangrams and has to be reassembled into another one. The goal is to have simple activities where the intuition of area can be developed without the distraction of measurement.

Activity 1. Here is a rectangle made with tangrams:



Form a parallelogram

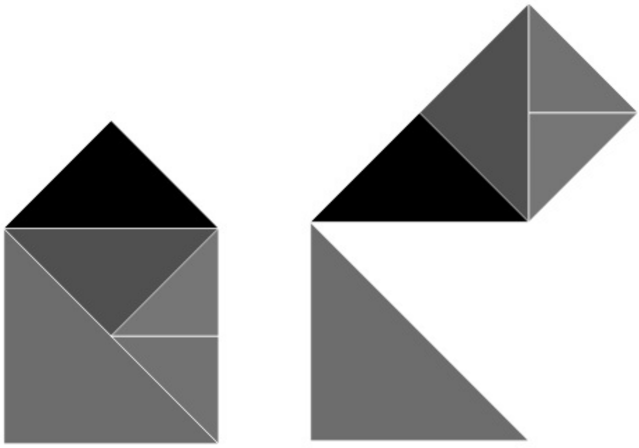
and a 'house' shape



with the same area.

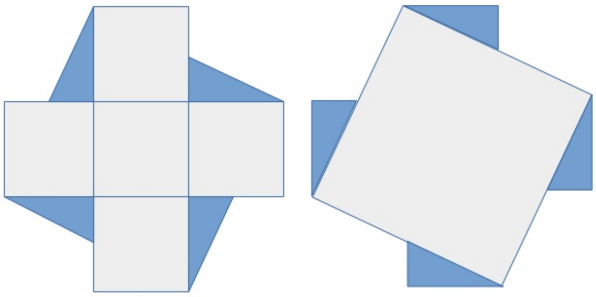
Through such activities, the student can get accustomed to the manipulations that later yield the area formulas for various polygons.

Tangrams can also give a simple demonstration of the Pythagoras Theorem for the case of an isosceles right triangle, if we allow ourselves to use an extra piece of the middle-sized triangle. We first make a square on the hypotenuse of the black triangle as shown. Then the three triangles making one half of this square are moved to form a square on one of the other sides.



It's worth noting that several websites enable one to manipulate tangrams online. The Mathigon website [10] has an especially nice implementation.

"Thinking that can be called algebraic" comes more to the fore when we consider another fact related to area: Two polygons have the same area if and only if one can attach congruent shapes to them in such a way that the two final shapes are congruent. The following diagram illustrates this:



The two gray shapes have the same area. Area calculations based on this principle resemble the manipulations of the sides of an equation.

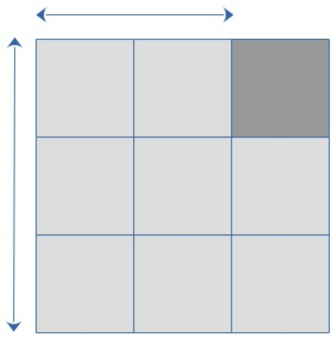
The two characterizations of polygonal area have an obvious direction and a non-trivial direction. For example, it is not obvious that any two polygons of the same area can definitely be cut into polygons and reassembled into each other. Remarkably, this was only established in the 19th century, independently by William Wallace, Farkas Bolyai and Paul Gerwien. Hilbert's Third Problem asked if volumes of polyhedra had similar behaviour and here the answer is negative! A cube cannot be cut into polyhedra which can be reassembled into a regular tetrahedron [1].

## 12.2 Squares in Mesopotamia

In this section, we'll describe some examples from Mesopotamia, from the "Old Babylonian period" of about 2000 to 1600 BC. This included the reign of Hammurabi, famous for his code of law. The mathematics of Old Babylon was initially thought to be algebraic due to its word problems, the step-by-step numerical instructions, and the tables of quantities like reciprocals and square roots. At the same time, as justifications were not provided, it was supposed that this algebra was based on empirical observations rather than reasoning. In the 1980s, new interpretations emerged from a closer reading of the Babylonian tablets and their contexts. The new viewpoint is that what was taken to be algebra is actually an expression of geometric insight and reasoning [6, 7].

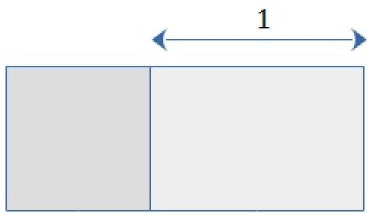
Here is an example from [7]. A rectangle is given with area 1/6 square units. Make two squares, one whose side equals the length of the rectangle, and another with side equal to the excess of the length over the width. Nine of the small squares fill the large square. What are the length and width of the original rectangle?

First, the large and small squares must make the following configuration.

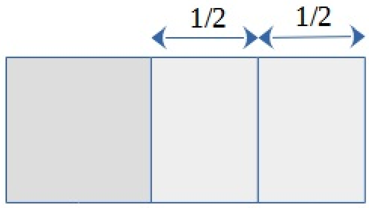


Since the side of the small square is the excess of the length over the width, the width must correspond to two squares, while the length corresponds to three. Hence the original rectangle consists of six of the small rectangles, in a $3 \times 2$ arrangement. It follows that each small square has area 1/36 and side 1/6. So the rectangle has length 1/2 and width 1/3.

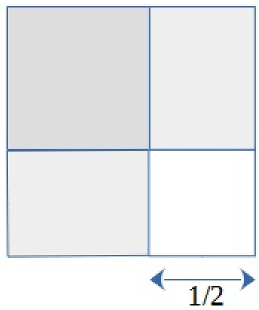
In a second example from [7], the area and side of a square are given to sum to 3/4, and the side of the square is to be found. This was one of the problems that led to the notion that the Old Babylonians were sometimes just playing abstractly with numbers, for what is the sense in adding an area to a length? However, it turns out that the 'side' is to be interpreted as the area of an attached rectangle with a unit side as shown below.




In the first step, the attached rectangle is bisected.




Then the outer rectangle is rotated and attached to the base of the original square, and the shape completed to a square.




The white square has area $1/4$, so now the total area is $3/4 + 1/4 = 1$. Hence the completed square has side 1, and the original square has side $1 - 1/2 = 1/2$.

This geometric procedure can be mapped to the 'completion of squares' that we learn in algebra. In algebraic notation, the problem at hand is to find a positive solution to $x^2 + x = 3/4$. We add $1/4$ to both sides so that the left side becomes a perfect square: $x^2 + x + 1/4 = 1$ or $(x + 1/2)^2 = 1$. This gives $x + 1/2 = 1$, hence $x = 1/2$.

With these two examples, we hope to make the point that simple geometric problems can lead to chains of reasoning which are based on visual intuition rather than manipulation of symbols, and that these can be chosen to prepare the ground for specific algebraic processes that are to be taught later.

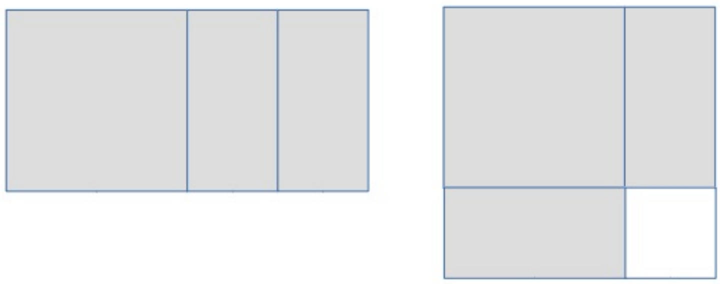
Completion of squares allows us to solve arbitrary quadratic equations, provided we can find square roots. Interestingly, it also yields a process for estimating square roots by rational numbers. While the Mesopotamians appear to have achieved this, we shall take up this story in the context of ancient India, where we find it mentioned more explicitly.

## 12.3 Square Roots in India

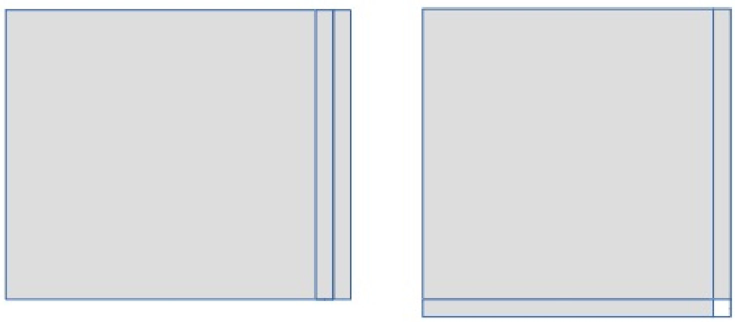
In India, the earliest clear descriptions of geometric facts are found in the Śulvasūtras. These texts address the concrete steps involved in the construction of the Vedic altars. Consequently, their instructions reveal aspects of the contemporary knowledge of geometry, especially as the area occupied by an altar was one of the issues of concern.

The 'completion of squares' process described above shows up in the Śulvasūtras in the treatment of the following question: Given a rectangle, make a square of the same area. Our process changes a rectangle to a square with an excess area which is also represented by a square. Thus the problem reduces to: Given two squares, make a square whose area is the difference of their areas. This is easily done with the help of Pythagoras' Theorem, whose statement is also found in these texts. (See Dani [3].)

Yet another perspective on this process is that conversion to a square with small excess area corresponds to finding an approximate square root. Thus this process can be used to generate rational approximations to square roots. For example, let us apply this process to a rectangle whose sides are of 1 and 2 units:



We get a square whose sides are 3/2 and area is in excess of the original. Thus 3/2 is an over-estimate of $\sqrt{2}$. Let us repeat the process starting with a rectangle of area 2 and with one side of 3/2. Therefore its other side is 4/3.



This new figure has an inner square of side 4/3 and a correction term of $\frac{1}{2}\left(\frac{3}{2}-\frac{4}{3}\right)=\frac{1}{12}$. Thus the new estimate is $\frac{4}{3}+\frac{1}{12}=\frac{17}{12}$. The Śulvasūtras express this in a convenient way: $1+\frac{1}{3}+\frac{1}{3\cdot 4}$ .

Let's carry out this process one more time, with a rectangle of sides 17/12 and 24/17 so that its area is 2. The rectangles created by halving the excess of 17/12 over 24/17 each have a width of $\frac{1}{2}\left(\frac{17}{12}-\frac{24}{17}\right)=\frac{1}{408}$. This time we view the correction as a removal of 1/408 from the initial 17/12 to get

$$\sqrt{2}\approx\frac{17}{12}-\frac{1}{408}=\frac{577}{408}=1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34},$$

and the last expression is again found in the Śulvasūtras!

This process can of course be expressed algebraically, and can be used for any $\sqrt{N}$ estimation. Start with a rectangle of sides $a, b$ such that $a > b$ and $ab = N$. The resulting square then has side $a' = b + \frac{a-b}{2} = \frac{1}{2}(\frac{N}{a} + a)$. This gives the iteration $a_{n+1} = \frac{1}{2}(\frac{N}{a_n} + a_n)$, starting with any convenient $a_1$. This process is called Heron's method after the Greek mathematician who mentioned it in passing while discussing the square root calculations required by his area formula. It is also what we get if we apply the Newton-Raphson formula to this problem! Finally, the Bakhshali manuscript, the only surviving physical evidence of ancient Indian mathematics, uses a scheme for square root calculations which consists of a double application of Heron's formula. Thus, this particular bit of history is rich with connections across time and space.

The references given below can yield further examples of geometric approaches that can be precursors to the teaching of algebra in the classroom. The article by Shirali [13] in fact has several such explorations with the Bakhshali manuscript as a starting point and the connections extending to continued fractions and Maclaurin series.

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# 13 Determination of the value of $\pi$ with the help of physical measurement by Rabindranath Chattopadhyay

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Abstract

On the basis of homogeneity of bond-paper sheets, different geometrical areas, initially circumscribed by a circle and then cut out from it, are weighed separately on a digital analytical balance. These measurements are combined with area formulas to obtain estimates of $\pi$.

## 13.1 Introduction

Attempts to estimate $\pi$ in the classroom are hampered by the difficulty of accurately measuring areas and curved lengths. On the other hand, if we have material of uniform density then we can use weight as a proxy for area and can obtain much more accurate values. In this article, we describe a classroom activity that uses standard A4 bond paper for this purpose.

The following shapes and corresponding area formulas are used:

1. Area of an equilateral triangle having its arms equal to $a$ is given by $A_t = (\sqrt{3}/4)a^2$ where $a = r\sqrt{3}$, $r$ being the radius of the corresponding circumcircle.

2. Area of a square with arms equal to $a$ is $A_s = a^2$ where $a = r\sqrt{2}$, r being the radius of the corresponding circumcircle.

3. Area of a regular hexagon having each arm equal to $a$ is given by $A_h = (3\sqrt{3}/2)a^2$ where $a = r$, $r$ being the radius of the corresponding circumcircle.

These lead to the following ratios of encircled area to the area of corresponding circle ($\epsilon_{-c}$) and also of the encircled area subtracted from the area of the corresponding circle (residual area) to the encircled area ($\eta_{-c}$):

1. $\epsilon_{tc} = \dfrac{3\sqrt{3}}{4\pi}$ and $\eta_{tc} = \left(\dfrac{4\pi}{3\sqrt{3}} - 1\right)$ for equilateral triangle.

2. $\epsilon_{sc} = \dfrac{2}{\pi}$ and $\eta_{sc} = \left(\dfrac{\pi}{2} - 1\right)$ for square.

3. $\epsilon_{hc} = \dfrac{3\sqrt{3}}{2\pi}$ and $\eta_{hc} = \left(\dfrac{2\pi}{3\sqrt{3}} - 1\right)$ for hexagon.

## 13.2 The Experiment

1. We began by weighing a few A4-size bond paper sheets on an electronic balance to establish that their weights were more or less the same [Table I].

2. Then on those papers five circles of same radius were drawn and cut out with the help of a sharp cutter very carefully and accurately.

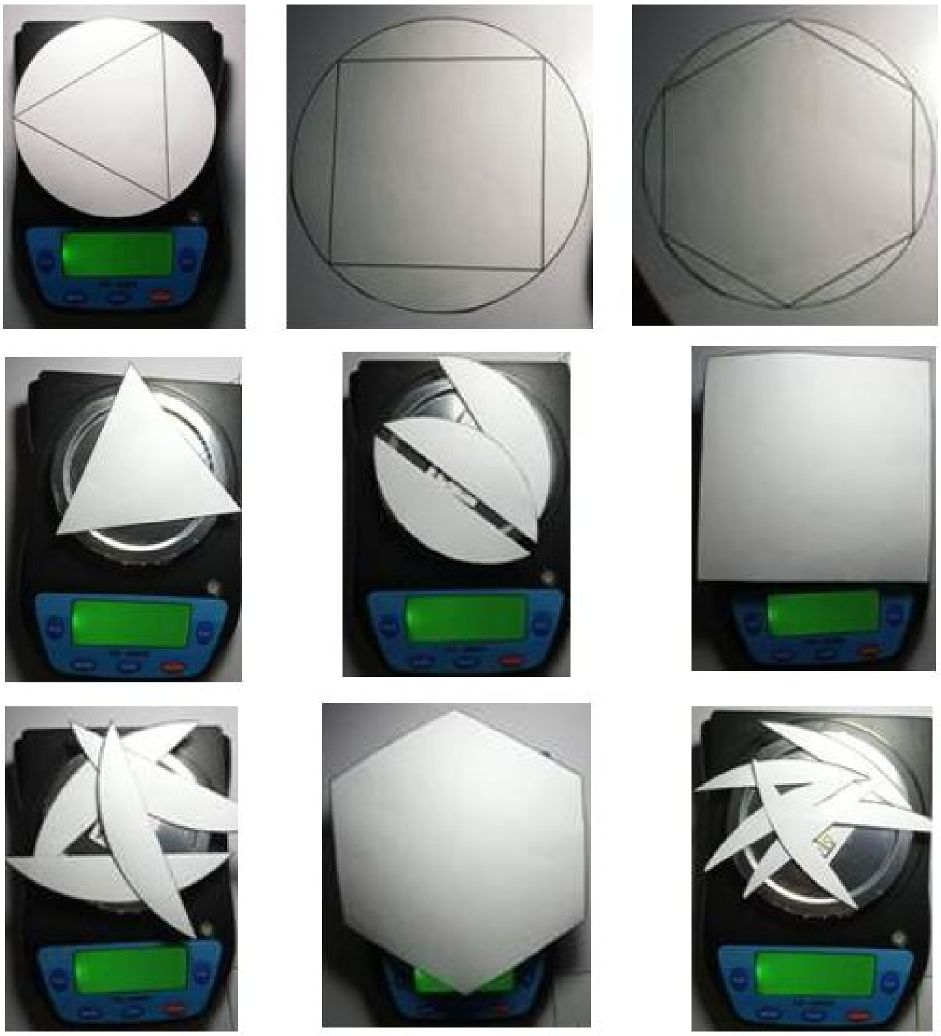
3. Two equilateral triangles were inscribed within two such circles. (Draw a diameter of the circle, then construct two 30° angles on each side of the diameter at the same endpoint. Extend the arms of the angles till they intersect the circle. The two points of intersection are then joined with straight lines.)

4. Two squares were inscribed within two more circles. (Draw two mutually perpendicular diameters of the circles. Connect the four points of intersection with the circular periphery with straight lines.)

5. Within the remaining circle a regular hexagon was inscribed whose vertices were all on the periphery of the circle. (A diameter is drawn first and then 60° angles are drawn on both sides at both ends of the diameter. The arms of the angles intersect the circle at four different points which are then connected with straight lines.)

6. The circles were assigned serial numbers 1, 2, 3, 4 and 5 to distinguish one from another. The circles were weighed with the help of a digital analytical balance and the weights were recorded against each serial number in tabular form. Then all the triangles, squares and hexagon were cut out from the circumscribing circular areas very carefully and again some separate distinguishing marks were put on each area to identify which area and residual area are cut from which circle. All these areas then were weighed separately on analytical balance as shown in the figures below.

7. The residual portion of the circular areas after removal of different geometrical figures were also assigned distinguishing mark and weighed again similarly and separately in respective cases.



For convenience of recording data in a tabular form and of calculation thereafter the following details were framed;

$$\epsilon_{tc} = \text{ ratio of weights of triangle to circle} = \frac{w_{tc}}{w_c} = \frac{\omega A_{tc}}{\omega A_c} = \frac{3\sqrt{3}}{4\pi}$$

where $\omega$ = weight of unit area of the paper (Here the homogeneity of all paper-sheets is presumed). $A_{tc}$ and $A_c$ are the area of triangle and the corresponding circle respectively. Then one can write

$$\pi = \frac{3\sqrt{3}}{4\epsilon_{tc}}$$

and $\eta_{tc}$ = ratio of the weights of triangular residual area and of triangular area $= \dfrac{w_c - w_{tc}}{w_{tc}} = \dfrac{4\pi}{3\sqrt{3}} - 1$ and $\pi = \dfrac{3\sqrt{3}}{4}(\eta_{tc} + 1)$.

Similarly for squares and hexagon one may write the following equations for $\pi$:

For squares, $\epsilon_{sc} = \dfrac{w_{sc}}{w_c} = \dfrac{\omega A_{sc}}{\omega A_c} = \dfrac{2}{\pi}$ and $\pi = \dfrac{2}{\epsilon_{sc}}$ and $\eta_{sc} = \dfrac{w_c - w_{sc}}{w_{sc}} = \dfrac{\pi}{2} - 1$ giving $\pi = 2(\eta_{sc} + 1)$ .

For regular hexagon, $\epsilon_{hc} = \dfrac{w_{hc}}{w_c} = \dfrac{\omega A_{hc}}{\omega A_c} = \dfrac{3\sqrt{3}}{2\pi}$ and $\pi = \dfrac{3\sqrt{3}}{2\epsilon_{hc}}$. Further, $\eta_{hc} = \dfrac{w_c - w_{hc}}{w_{hc}} = \dfrac{2\pi}{3\sqrt{3}} - 1$ gives $\pi = \dfrac{3\sqrt{3}}{2}(\eta_{hc} + 1)$.

## Sample Observations and Results

| Shapes / Parameters | $A_4$ size papers | | | | | Circular cuts | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (1) | (2) | (3) | (4) | (5) |
| Area in Sq.Cm. | 623.7 | 623.5 | 623.7 | 625.2 | 626.2 | 176.6 | 174.9 | 172.2 | 170.7 | 169.5 |
| Weights in gm.wt. | 4.26 | 4.24 | 4.25 | 4.26 | 4.25 | 1.18 | 1.16 | 1.12 | 1.07 | 1.06 |

Table 1: Measured values of areas (from lengths and diameters) and weights of A4 size papers and corresponding circular cuts.

| Shapes / Parameters | Equilateral Triangle(1) | Equilateral Triangle(2) | Square(3) | Square(4) | Regular Hexagon(5) | Mean Value of π |
|---|---|---|---|---|---|---|
| $w_c$ | 1.18gm | 1.16gm | 1.12gm | 1.07gm | 1.06gm | 3.1389 |
| $w_{tc}/w_{sc}/\, w_{hc}$ | 0.49gm | 0.48gm | 0.71gm | 0.68gm | 0.88gm | |
| $w_c - (w_{tc}/w_{sc}\, / \, w_{hc})$ | 0.69gm | 0.68gm | 0.41gm | 0.39gm | 0.18gm | |
| $\varepsilon_{tc}/\varepsilon_{sc}/\, \varepsilon_{hc}$ | 0.4150 | 0.4138 | 0.6339 | 0.6355 | 0.8306 | |
| $\eta_{tc}/\eta_{sc}/\, \eta_{hc}$ | 1.4082 | 1.4167 | 0.5770 | 0.5735 | 0.2045 | |
| π (ε ) | 3.130 | 3.139 | 3.150 | 3.147 | 3.130 | |
| π (η ) | 3.128 | 3.139 | 3.150 | 3.147 | 3.129 | |

Table 2: Experimental data and values of the corresponding parameters in five cases.

The mean value of the constant $\pi$ as obtained from the above table(Table 2) containing observed values and calculated values of different variables and parameters respectively in five different cases is found to be '3.1389' which is very close to the actual value '3.1416'. Due to observational and systematic errors that might have crept in the whole process of measurement and determination, slight deviation from the actual value might have occurred. Further repetition of this experiment by a large number of other observers could reduce the deviation to reach more accurate value of $\pi$. Again on the other side of the experiment the assumption of homogeneity of the bond paper might not hold in reality and for that reason too deviations might have occurred.