Clarifications for Students' Queries

Queries were received on questions 7, 9, 19, 21, 23 and 27. We have given below detailed clarifications to these queries. Some other queries on correctness of answers for a few other questions were received. We confirm that the answers given in our web-site are correct. The students who have such concerns can re-check their solutions themselves.

7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is coloured either red or yellow with opposite faces always of the same colour. The dice are distinguished by the colour pattern as well as the numbering pattern. Find the number of distinct dice that can be designed?

   The solutions submitted claiming that the answer must be 24 are incorrect.

   Since 1 and 2 are always opposite, the arrangement of 3,4,5,6 decides the number pattern. Each such arrangement corresponds to a circular permutation of 3,4,5,6. Hence there are $3! = 6$ such arrangements. Also, each pair of opposite faces have the same colour and there are 2 colours for use. Thus, the number of distinct designs is $6 \times 2^3 = 48$.

9. Find the number of triples (a, b, c) of positive integers such that

   (a) ab is a prime;

   (b) bc is a product of two primes;

   (c) abc is not divisible by square of any prime and

   (d) $abc \leq 30$.

   The query was that whether the triples are ordered or not is not specified. But we are counting triples (a, b, c) that satisfy the conditions and the notation clearly means that the triples are ordered.

   The possible triples are

$$
\begin{array}{ccccc}
(1,2,3) & (1,2,5) & (1,2,7) & (1,2,11) & (1,2,13) \\
(1,3,2) & (1,3,5) & (1,3,7) & (1,5,2) & (1,5,3) \\
(2,1,15) & (3,1,10) & (5,1,6) & (1,7,2) & (1,7,3) \\
(1,11,2) & (1,13,2) & & &
\end{array}
$$

   We have a total of 17 triples.

19. For $n \in \mathbb{N}$, let $P(n)$ denote the product of the digits in n and $S(n)$ denote the sum of the digits in n. Consider the set

$$A = \{n \in \mathbb{N} : P(n) \text{ is non-zero, square free and } S(n) \text{ is a proper divisor of } P(n)\}.$$

Find the maximum possible number of digits of the numbers in A.

The query was that the term square-free is not understood. But square-free is a standard term in Number Theory. A number is square-free if it is not divisible by the square of any prime number.

21. For $n \in \mathbb{N}$, consider non-negative integer valued functions f on $\{1, 2, \ldots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n}(i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

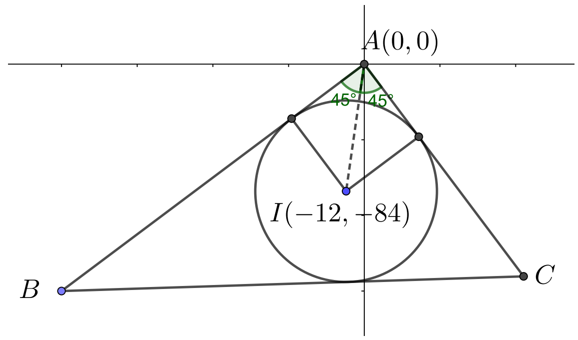
More than 90% of the queries were pertaining to this question claiming that there is ambiguity in the statement. It was argued that a function f *defined on* S meant that the domain and co-domain are both S. We have not seen such a definition of function anywhere. Also, surprisingly, almost all the mails on this question had the same text in the body of the mail.

However, there is **NO AMBIGUITY** in the statement. We are considering functions that are non-negative integer valued – hence functions that take values in non-negative integers (so that the co-domain is the set $\{0, 1, 2, \ldots\}$). The function is defined on $\{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Thus the domain of the function is the set $\{1, 2, \ldots, n\}$. When we say a function is defined **on** a set S, by standard convention, this means that the domain of the function is S. Note that it need not take values in the same set. When we say that a function **takes values in a set** T, then by standard convention, it means that the codomain of the function is the set T.

23. In the coordinate plane, a point is called a *lattice point* if both of its coordinates are integers. Let A be the point $(12, 84)$. Find the number of right angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin $(0, 0)$.

It was claimed that the answer should be 24. But the solution presented by the students were erroneous. The correct solution is given below.

We will shift the vertex A to the origin. We need to find the number of triangles with vertices B, C at lattice points and incenter at $(-12, -84)$. The slope of AI



is $84/12 = 7$. If m is the slope of AC, then, since $\angle IAC = 45°$, we have

$$\tan 45° = \frac{m-7}{1+7m} \Rightarrow m = -\frac{4}{3}$$

Since $AB \perp AC$, slope of AB is $\frac{3}{4}$. Hence we can write B as $(-4t, -3t)$ and C as $(3t', -4t')$ for some positive t, t'. Since B and C are lattice points, it follows that t, t' are integers.

The inradius of the triangle ABC is $r = AI \cos 45° = 60$. Also, we have $BC = AB + AC - 2r$. Thus

$$5\sqrt{t^2 + t'^2} = 5t + 5t' - 120$$
$$\Rightarrow tt' - 24(t + t') + 288 = 0$$
$$\Rightarrow (t - 24)(t' - 24) = 288 = 2^5 \cdot 3^2$$

Thus $t - 24$ can be any divisor of 288 and there are 18 divisors for 288.

The only possible negative values for $t - 24$ and $t' - 24$ are $(-16, -18)$ and $(-18, -16)$. It is easy to see that the resulting triangle for these choices do not contain $(-12, -84)$ as in-center. Also, such a choice violates the condition $BC = AB + AC - 2r$. Thus we need to count only the positive factors of 288. There are 18 divisors for 288 and hence there are 18 such triangles.

27. A quadruple (a, b, c, d) of distinct integers is said to be *balanced* if $a + c = b + d$. Let $\mathcal{S}$ be any set of quadruples (a, b, c, d) where $1 \leq a < b < d < c \leq 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples in $\mathcal{S}$.

It was claimed that the term cardinality was not defined. However, we maintain that cardinality of a set is a common term defined in set theory. Even the dictionary definition is "number of members in a set or a group".